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Planar Quasiconformal Mappings

Fundamental Properties and Characterizations

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ABSTRACT OF
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Quasiconformal (QC) mappings generalize conformal mappings. Since their introduction in the 1930s, QC mappings have become a versatile tool in various fields of mathematics, ranging from PDEs to holomorphic dynamics.

This thesis is an exposition of the five most widespread descriptions of QC mappings in the plane, as well as the most valuable properties thereof. We present a proof of the equivalence of the three main definitions: the metric, analytic, and geometric. Two additional characterizations are discussed in detail. The first is the partial identification of QC mappings with quasisymmetric mappings. This is done via conformal invariants. Once this identification is obtained, we use it to demonstrate that QC maps form a pseudogroup. We also use quasisymmetries to obtain the compactness properties of certain families of QC maps. Further, we demonstrate, using complex variables, several analytic properties, such as the change of variables and area formulæ. We present a proof of the Measurable Riemann Mapping Theorem, which identifies quasiconformal mappings as the solutions of the Beltrami's equation — this is the fifth characterization.

It is the interplay between the alternative characterizations that is arguably the most prominent feature of QC mappings. For this reason, an emphasis is put on highlighting the relationships between various descriptions and approaches to proofs.

Keywords:	quasiconformal, quasisymmetric, distortion, homeomorphism, Beltrami equation, Measurable Riemann Mapping Theorem, conformal invariant, ring capacity, symmetrization
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Foreword

Purpose of the Work

This thesis is an exposition of the foundations of the theory of quasiconformal (henceforth QC) mappings in the planar setting, aimed at the beginner to intermediate level graduate students. The theory of QC mappings has been in extensive development since the late 1920s and now comprises a vast body of mathematical techniques. This theory enjoys an immensely broad range of applications: from PDEs and inverse problems to holomorphic dynamics, Teichmüller theory, and image manipulation.

Despite minimal assumptions and relatively unsophisticated definitions, the class of QC mappings possesses remarkable regularity properties, which allow much of the calculus to be performed on it. This class retains flexibility when extended to higher dimensions, unlike conformal mappings, and possesses useful compactness properties. Quasiconformal mappings are homeomorphisms, they form a pseudogroup, and exhibit almost affine behaviour on local scales. These are only some of the properties that make this class such a useful tool in various fields.

However, such versatility may be a hindrance for the uninitiated. The abundance of the diverse characterizations can at times make it difficult to build a coherent bird's-eye perspective on the field. Most popular texts and surveys are massive monographs that employ heavy mathematical machinery. It is not always clear where to begin the study of QC mappings, given the variety of equivalent definitions.

This thesis is an attempt at a digest of the five most widely used characterizations of QC mappings. It is an experiment at drawing parallels between different approaches to the topic. An endeavor in which, admittedly, rigour occasionally gives way to insight.

Content and Arrangement

The text begins with an introductory chapter in which several basic definitions and results from early graduate-level analysis courses are recalled. Another purpose of this chapter is to agree upon notation. The prerequisites include a working knowledge of the Lebesgue integration theory, Sobolev spaces, and basic complex function theory. Some more extensive reminder on the latter is provided also in Chapter 4.

Roughly speaking, QC mappings are mappings of bounded distortion; in this context, distortion is a measure of how far from being conformal a given mapping is. Chapter 1 introduces three ways to measure distortion: metric, analytic, and geometric. The first is the most accessible, the second requires some real analysis and Sobolev spaces. The geometric distortion is based on a conformal invariant, namely the ring capacity.

Chapter 2 is all about proving the equivalence of the three definitions of quasiconformality that arise from the different notions of distortion.

Chapter 3 introduces a fourth characterization of QC mappings via the concept of a quasisymmetric function. Simply put, quasisymmetries play a similar role to QC mappings as similarities (mappings preserving congruence of shapes) play to conformal maps. In particular, every similarity is QC, which we show via the metric definition. On the other hand, a QC map restricts to a quasisimilarity on a sufficiently small domain. The proof in this direction is carried out by using the geometric definition; to this end, we evoke an isoperimetric-type inequality for the ring capacity.

Until this point, most of the discussion is based on real analysis; Chapter 4 starts the transition to the complex setting. Another, fourth, definition of distortion is presented, this time in terms of a complex function called dilatation. An important lemma of Weyl identifies a particular subclass of QC mappings with conformal maps.

In Chapter 5, we collect several fundamental properties of QC mappings. We show, via the identification with quasisymmetries, that inverses and compositions of QC maps are QC. We discuss integral formulæ, namely the change of variables and the area formula, as well as the chain rule. Finally, we obtain important compactness properties of the classes of QC maps. All these properties are essential for what comes in the final chapter.

Finally, in Chapter 6, we prove the Measurable Riemann Mapping Theorem, thus establishing that planar QC mappings can be identified with the solutions of a certain PDE, named after Eugenio Beltrami. This chapter is arguably the heaviest one to digest, mostly because it evokes advanced machinery such as singular integral transforms of Cauchy and Beurling.

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Contents

Preliminaries and Notation

In this chapter, we set up notation and terminology as well as recall some basic preliminary results used throughout this text. Most of the topics discussed here are covered in detail in basic undergraduate and first year graduate level courses. For further reference, one may consult [Rud76], [Rud87], [Kin21].

The Euclidean plane. Conformal and quasiconformal mappings were historically first explored in the complex setting. The modern treatment of the subject, however, is real-analytic at its core — such approach affords straightforward generalizations of many concepts to higher dimensions.

The bulk of our exposition in Chapers 1–3 is set in the *Euclidean plane* $\mathbb{R}^2$. A generic element of the plane x is uniquely described by an ordered pair of real coordinates, $x = (x_1, x_2)$.

We equip the plane with the usual *2-norm*:

$$|x| = \sqrt{x_1^2 + x_2^2}.$$

The norm naturally induces the *distance function* $\operatorname{dist}(x, y) = |x - y|$. We use the following notations for the *open disc*, *closed disc*, and *circle* (all of radius $r > 0$ with centre at x):

$$\begin{aligned} \mathbb{D}_r(x) &= \left\{ y \in \mathbb{R}^2 \colon |x - y| < r \right\}, \\ \overline{\mathbb{D}}_r(x) &= \left\{ y \in \mathbb{R}^2 \colon |x - y| \leqslant r \right\}, \\ \mathbb{S}_r(x) &= \left\{ y \in \mathbb{R}^2 \colon |x - y| = r \right\}. \end{aligned}$$

We also employ the following shorthands:

$$\mathbb{D}_r = \mathbb{D}_r(0), \qquad \mathbb{D} = \mathbb{D}_1, \qquad \overline{\mathbb{D}}_r = \overline{\mathbb{D}}_r(0), \qquad \overline{\mathbb{D}} = \overline{\mathbb{D}}_1.$$

The *diameter* of a set A and the *distance between sets* A, B are defined by

$$\operatorname{diam}(A) = \sup\left\{|x-y|\colon x, y \in A\right\},$$
$$\operatorname{dist}(A, B) = \inf\left\{|x-y|\colon x \in A,\ y \in B\right\}.$$

The Euclidean distance induces the standard topology on the plane; the symbols A°, $\bar{A}$, and ∂A stand for the *interior*, *closure*, and *topological boundary* of $A \subset \mathbb{R}^2$, respectively. We say that A is *compactly contained* in B and write $A \subset\subset B$ whenever $\overline{A} \subset B$.

The Complex plane. We identify $\mathbb{R}^2$ with the *complex plane* $\mathbb{C}$ via the rule

$$\mathbb{R}^2 \ni (x, y) \;=\; z \;=\; x + iy \in \mathbb{C},$$

where i is the *imaginary unit* defined by $i^2 = -1$. In this notation, we regard the reals x and y as the *real* and *imaginary part* of the complex number z. Complex variables will play a crucial role in the final chapter, where we discuss exclusively planar phenomena.

We define the *complex conjugate* $\bar{z}$ of $z = x+iy$ by $\bar{z} = x-iy$. In view of this, we have $|z| = \sqrt{z\bar{z}}$. Every complex number can be written conveniently in its *polar form* $z = |z|e^{i\operatorname{Arg}(z)}$, where $\operatorname{Arg}: \mathbb{C} \to [0, 2\pi)$ is the *argument function.*

We recall also that the plane admits a compactification $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ called the *extended plane.* Further, $\overline{\mathbb{C}}$ is identified with the *Riemann sphere* by means of the stereographic projection, which also endows $\overline{\mathbb{C}}$ with a metric and a topology. This construct is particularly helpful when used with *Möbius transformations*;[1] these are the functions of the form

$$z \mapsto \frac{az+b}{cz+d},$$

where a, b, c, d are complex numbers such that $ad - bc \neq 0$. Every Möbius transformation is a composition of translation, rotation, homothety (scaling), and inversion of $\mathbb{C}$. Further, once we adopt the standard conventions

$$\frac{1}{0} \mapsto \infty, \qquad \frac{1}{\infty} \mapsto 0, \qquad 1 \cdot \infty \mapsto \infty,$$

we can regard Möbius transformations as transformations of the Riemann sphere $\overline{\mathbb{C}}$. One appplication of this is *normalization* of planar mappings. Because every Möbius transformation is a homeomorphism of the sphere $\overline{\mathbb{C}}$,

[1] Also known as *linear fractional transformations.*

we can study the behaviour of a map $f\colon \mathbb{C} \to \mathbb{C}$ near 'bad' points (such as poles of f or the point ∞) by examining the *conjugate* map $g = \varphi^{-1} \circ f \circ \varphi$, where φ is a suitable Möbius transformation. We discuss this in more detail in §4.3.

Linear transformations and similarities. Recall that the *determinant* is the unique function

$$\det\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

that is multilinear, antisymmetric, and assigns the value 1 to the *canonical basis* e_1, e_2 of $\mathbb{R}^2$. The determinant of two vectors is zero if and only if they are linearly dependent.

The *determinant of a planar linear transformation* T is defined by

$$\det(T) = \det(Te_1, Te_2).$$

We recall that a transformation is *singular* or *noninvertible* if and only if its determinant is zero. Further, the determinant is multiplicative on the class of linear maps, that is, $\det(S \circ T) = \det(S)\det(T)$ for all linear S and T.

Throughout our discussion, we need to keep a geometric interpretation in mind: the determinant of a linear transformation is the multiplicative factor by which the transformation changes (signed) areas of sets in $\mathbb{R}^2$.

We say that a linear transformation is *sense-preserving* if it preserves the sign of the inner product.

A linear isometry of $\mathbb{R}^2$ is called an *orthogonal transformation*; such transformations, being isometries, preserve the inner product up to a sign change and have the determinant either $+1$ or -1. Sense-preserving orthogonal transformations — those with unit determinant — are referred to as *rotations*.

We call a planar mapping S a *similarity transformation* (or *similarity* for short) if it has the form

$$S(x) = \lambda O x + b$$

for some $\lambda > 0$, a rotation O, and $b \in \mathbb{R}^2$. In plain language, a similarity is a composition of a scaling, rotation, and translation; it maps geometric shapes to congruent (similar) shapes. Multiplicativity of the determinant entails that $|\det(\lambda O)| = |\det(\lambda I)| \cdot |\det(O)| = \lambda^2$.

We appoint the symbol $\|T\|$ for the *operator norm* of a linear transformation T which is defined by

$$\|T\| = \sup_{|x|=1} |Tx|.$$

Domains, homeomorphisms, mappings. By a *domain* we mean an open connected set. In this text, the symbol Ω will always denote a generic planar domain.

In our exposition, we reserve the term *mapping* (or *map*) for functions with values in $\mathbb{R}^2$. That is, the phrase 'f is a mapping on A' is to be interpreted as '$f\colon A \to \mathbb{R}^2$.' We shall say '$f$ is a mapping *of* A' in the case when f transforms A onto itself: $f(A) = A$. The term *function* is used in its full generality.

A *homeomorphism* is a continuous function, whose inverse exists and is also continuous. In particular, a homeomorphism is necessarily one-to-one (injective) and onto its image. Homeomorphisms preserve topological invariants: openness and closedness, connectedness, set inclusions, etc. etc. In particular, the image of a domain under a planar homeomorphism is itself a domain, the image of a simply (resp. doubly) connected set is simply (resp. doubly) connected, and so forth.

In our exposition, we are only concerned with planar homeomorphisms. That is, when we say 'f is homeomorphic in Ω' we mean '$f\colon \Omega \to \Omega'$, where Ω and Ω' are domains in the plane.'

Differentiability. Let f be a function defined on a planar domain Ω and let x be a point in Ω. The quantity defined by the limit

$$\partial_h f(x) = \lim_{\mathbb{R}\ni t\to 0} \frac{f(x+th) - f(x)}{t},$$

if it exists, is called the *directional derivative* of f (at x) associated with vector h. The *partial derivatives* (or *partials* for short) are the directional derivatives associated with the canonical basis vectors; we use the symbol $\partial_i f(x)$ to denote the i-th partial derivative of f at x. In the case when f is real-valued and both partials exist, we define the *gradient (vector)* of f at x by

$$\nabla f(x) = \big(\partial_1 f(x),\ \partial_2 f(x)\big)\,.$$

We say that f is *differentiable*[2] at x, if there is a linear transformation

[2]in the sense of Fréchet.

$Df(x)$ such that

$$\lim_{\mathbb{R}^2 \ni h \to 0} \frac{|f(x+h) - f(x) - Df(x)\,h|}{|h|} = 0. \tag{1}$$

If $Df(x)$ exists, we can approximate f in a neighbourhood of x by writing

$$f(x+h) = f(x) + Df(x)\,h + |h|\,\varepsilon(x,h), \tag{2}$$

where $\varepsilon(x,h) \to 0$ as $h \to 0$. Whenever this is the case, the linear transformation $h \mapsto Df(x)\,h$ is called the *differential* of f at x, and the affine transformation $h \mapsto f(x) + Df(x)\,h$ is called the *linearization* of f at x. The differential of f, when it exists, is determined uniquely by its action on an arbitrary vector h, since (1) entails

$$Df(x)\,h = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}. \tag{3}$$

A function has all directional derivatives at a point of differentiablility; in particular, $Df(x)\,h = \partial_h f(x)$ by virtue of (3). The converse is not true in general.

We say that f is differentiable in Ω, if it is differentiable at every point in Ω.

Suppose that f is differentiable at x. If f is real-valued, then the differential of f is the transformation $h \mapsto \nabla f(x) \cdot h$, where the dot stands for the Euclidean inner product. If f is a mapping with real components $f^{(1)}$ and $f^{(2)}$, the differential $Df(x)$ admits a (square) matrix representation

$$Df(x) = \begin{bmatrix} \partial_1 f^{(1)}(x) & \partial_2 f^{(1)}(x) \\ \partial_1 f^{(2)}(x) & \partial_2 f^{(2)}(x) \end{bmatrix},$$

called the *derivative matrix.*[3] Note that we use the same symbol to denote both the differential and its derivative matrix.[4] The *Jacobian determinant* or, simply, the *Jacobian* of f at x is defined by

$$Jf(x) = \det Df(x).$$

We say that a mapping is *sense-preserving* if its Jacobian determinant is positive and *sense-reversing* otherwise. Bearing in mind what the determinant of a linear transformation means geometrically, we interpret $|Jf(x)|$ as

[3] Also known as the *Jacobian matrix*.

[4] The reader most likely knows that the terms *derivative* and *differential* are used interchangeably throughout mathematics.

the factor by which f changes infinitesimal areas near x. It is for this reason that $|Jf(x)|$ appears in the area (4) and change of variables (5) formulæ below.

Finally, we recall the **the Chain Rule**. Let U and V be open subsets of $\mathbb{R}^2$. Suppose that $f\colon U \to \mathbb{R}^2$ is differentiable at a point $x \in U$ and $g\colon V \to \mathbb{R}^2$ is differentiable at $f(x) \in V$. Under these hypotheses, the composition $g \circ f : U \cap f^{-1}(V) \to \mathbb{R}^2$ is differentiable at x, and its derivative satisfies

$$D(g \circ f)(x) = (Dg \circ f)(x)\, Df(x).$$

For $h\colon V \to \mathbb{R}$ differentiable at $f(x) \in V$, the chain rule entails

$$\nabla(h \circ f)(x) = D^t f(x)\, (\nabla h \circ f)(x).$$

Differentiation of measures and integration. We use the symbol $|A|$ to denote the 2-dimensional (area) *Lebesgue measure* of a planar set A. The symbol $\mathcal{H}^1(A)$ stands for the 1-dimensional (linear) *Hausdorff measure* of the set A. It is a basic fact of measure theory that for every integer $n \geqslant 1$ the n-dimensional Lebesgue and the n-dimensional Hausdorff measures coincide.

Let μ be a nonnegative real-valued measure on $\mathbb{R}^2$. We define the *maximal derivative* of μ (with respect to the Lebesgue measure) at x by the limit

$$D^+\mu(x) = \limsup_{r \to 0} \frac{\mu(\mathbb{D}_r(x))}{|\mathbb{D}_r(x)|}.$$

The **Differentiation Theorem of Lebesgue** asserts that *if μ is a Borel measure, then the* symmetric derivative *of μ at x, defined by*

$$D\mu(x) = \lim_{r \to 0} \frac{\mu(\mathbb{D}_r(x))}{|\mathbb{D}_r(x)|},$$

exists and is finite almost everywhere. A closely related result is, of course, the **Radon–Nikodym Theorem**, which states that *if μ is a Borel measure, then $D\mu$ is locally absolutely integrable, and*

$$\int_B D\mu(x)\, dx \leqslant \mu(B) \qquad \text{for every Borel set } B,$$

with equality occuring if μ is absolutely continuous with respect to the Lebesgue measure. The latter is another way of saying that

$$|B| = 0 \quad \text{implies} \quad \mu(B) = 0 \quad \text{for all Borel sets } B.$$

Let f be a mapping defined on an open set E. It induces a set function ν_f, called the *pullback* of the area Lebesgue measure by f, on E via the rule

$$\nu_f(A) = |f(A)| \qquad \text{for } A \subset E.$$

The derivative $D\nu_f$ (when it exists) is called the *volume derivative* of f:

$$D\nu_f(x) = \lim_{r\to 0} \frac{\nu_f(\mathbb{D}_r(x))}{|\mathbb{D}_r(x)|} = \lim_{r\to 0} \frac{|f(\mathbb{D}_r(x))|}{|\mathbb{D}_r(x)|}.$$

Under the hypothesis that f is homeomorphic on E,[5] *the pullback ν_f is a Borel measure on E*, which follows from the definitions. Then, in light of the Radon–Nikodym Theorem, $D\nu_f \in L^1_{\mathrm{loc}}$, and

$$\int_B D\nu_f(x)\, dx \leqslant \nu_f(B) \qquad \text{for every Borel subset } B \subset E,$$

with equality if $|A| = 0$ implies $|f(A)| = 0$ for any Borel $A \subset E$. In other words, equality occurs if f preserves sets of measure zero. If this is the case, we say that f has the *Lusin's $\mathcal{N}$–property.*[6]

If f is not only homeomorphic but also differentiable at $x \in E$, then we have in fact

$$|Jf(x)| = D\nu_f(x).$$

For this reason, the volume derivative $D\nu_f$ has another name: the *generalized Jacobian* of f. Further, if f is an everywhere differentiable homeomorphism on E, then it preserves sets of measure zero, and we have an elementary version of **the area formula**:

$$|f(B)| = \int_B |Jf(x)|\, dx. \tag{4}$$

Lastly, for every measurable $h\colon \mathbb{R}^2 \to [0,\infty]$, **the change of variables formula**

$$\int_{f(X)} h(y)\, dy = \int_X (h\circ f)(x)\, |Jf(x)|\, dx \tag{5}$$

is valid under the assumption that f is continuous on E, as well as injective and differentiable on Lebesgue measurable $X \subset E$ satisfying $|f(E\setminus X)| = 0$. A detailed discussion of these results can be found in [Rud87, Chapter 7]. There are versions of above formulæ for almost everywhere differentiable functions (such as e.g. Lipschitz functions) and for Sobolev functions, see

[5] more generally, if f is a continuous injection onto its image

[6] $\mathcal{N}$ stands for *null*.

[EG15]. We shall see in Chapter 5 that the area and change of variables formulæ are also valid for quasiconformal mappings.

AC, ACL, and Sobolev functions. Let I be an interval on the real line. We say that a real function h is *absolutely continuous* (AC) on I, if it is continuous on I and for every $\epsilon > 0$ one can find a $\delta > 0$ such that every finite family of disjoint subintervals $\{(a_i, b_i)\}_{i=1}^k \subset I$ has the following property:

$$\text{if} \qquad \sum_{i=1}^k |a_i - b_i| < \delta, \qquad \text{then} \qquad \sum_{i=1}^k |h(a_i) - h(b_i)| < \epsilon.$$

This definition is quite a mouthful, and it does not involve planar maps. It extends to the following: a continuous mapping f is AC on a straight line segment L if for every $\epsilon > 0$ there is $\delta > 0$ such that

$$\mathcal{H}^1(J) < \delta \qquad \text{implies} \qquad \mathcal{H}^1(f(J)) < \epsilon$$

for every Borel subset $J \subset L$. In other words, f pulls back the linear measure $\mathcal{H}^1$ to a measure which is *absolutely continuous* with respect to $\mathcal{H}^1$. We recall that a real function is absolutely continuous if and only if Lebesgue's Fundamental Theorem of Calculus holds true for it.

A planar map is said to be *absolutely continuous on lines* (ACL) if its restriction to $\mathcal{H}^1$–almost–every line parallel to the coordinate axes is absolutely continuous. ACL functions are particularly nice to work with since they have partial derivatives almost everywhere.

Let $U \subset \mathbb{R}^2$ be open and let f be a function defined in U. We say that the function g is the *distributional* (or *weak*) *derivative* of f in U if the identity

$$\int_U g(x)\,\varphi(x)\,dx = -\int_U f(x)\,D\varphi(x)\,dx$$

holds for every smooth test function φ whose support is contained in U, $\varphi \in C_\circ^\infty(U)$. Recall that the *Sobolev class* $W^{1,p}_{\text{loc}}(U)$ consists of $L^p_{\text{loc}}(U)$ functions whose distributional derivatives also lie in $L^p_{\text{loc}}(U)$.

We use the **ACL characterization of the Sobolev class** on multiple occasions in this text. Let V be compactly contained in U, and suppose that $1 \leqslant p \leqslant \infty$. If $f \in W^{1,p}(U)$, then f is ACL on V and its classical partial derivatives coincide with the weak partial derivatives of f almost everywhere in U. Conversely, if f is of class $L^p_{\text{loc}}(U)$ and is ACL on V, then the partial derivatives of f exist almost everywhere, and $f \in W^{1,p}(U)$

provided the pointwise partials of f belong to $L^p_{\mathrm{loc}}(U)$. We remark that f is to be understood as a representative of the Lebesgue class, that is, we are allowed to modify it on sets of Lebesgue measure zero. In asserting this result, we cite Juha Kinnunen's lecture notes [Kin21].

Mollification of functions. The classical technique for extending results from smooth functions to Lebesgue-integrable or Sobolev functions is mollification. First, choose the *standard mollifier* — a C^∞ smooth 'bump' function φ with support in the unit disc such that $\|\varphi\|_{L^1(\mathbb{R}^2)} = 1$. Define the family of mollifiers by setting

$$\varphi_\epsilon(x) = \frac{1}{\epsilon^2}\,\varphi\left(\frac{1}{\epsilon x}\right)$$

for every $\epsilon > 0$. For a function f in $L^1_{\mathrm{loc}}(\Omega)$, the *mollifying family* is the collection of smooth functions $\{f_\epsilon\}_{\epsilon>0}$, every member of which is defined on the set

$$\Omega_\epsilon = \{x \in \Omega\colon\ \mathrm{dist}(x, \partial\Omega) > \epsilon\}$$

by the convolution with a mollifier:

$$f_\epsilon(x) = (f * \varphi_\epsilon)(x) = \int_\Omega f(y)\,\varphi_\epsilon(x-y)\,dy, \qquad x \in \Omega_\epsilon.$$

Mollification allows us to extend many calculus results to functions that lie in Lebesgue or Sobolev spaces. We record the following properties of convolution approximation, citing [AIM08].

Assume $f \in L^p_{\mathrm{loc}}(\Omega)$. There exists a sequence of functions $f_n \in C^\infty_\circ(\Omega)$ such that

- (i) *$f_n(z) \to f(z)$ for almost every $z \in \Omega$;*
- (ii) *the distributional derivatives satisfy $\|Df_n - Df\|_{L^p(\Omega')} \to 0$ for every compact subset $\Omega' \subset \Omega$;*
- (iii) *if $f \in C(\Omega)$, then $f_n \to f$ uniformly on compact subsets of Ω;*
- (iv) *if $f \in C(\overline{\Omega})$, then $f_n \to f$ uniformly.*

Chapter 1

The Concept of Distortion

As the name suggests, quasiconformal mappings are mappings that are 'almost conformal.' Therefore, in order to define quasiconformality, we need to devise ways of gauging how far from being conformal a given mapping is. This is done by measuring the distortion exhibited by the mapping. In this chapter, we introduce three approaches to measuring distortion: metric, analytic and geometric. These give rise to three equivalent definitions of a quasiconformal mapping, which are the topic of the next chapter.

In defining the metric and geometric distortion we follow [GMP17]. The metric distortion and its analytic properties are discussed in full detail in the monograph [IM01]. The geometric notion of distortion rests upon the so-called conformal invariants. For a comprehensive background in conformal invariants we refer the reader to [Ahl10] and [DK14].

We remind the reader that we only study sense-preserving homeomorphisms between planar domains in this work.

Metric Distortion of Infinitesimal Discs

1.1 **Definition.** Let f be a homeomorphism defined on a planar domain Ω. The *infinitesimal distortion* of f is a function $Hf\colon \Omega \to [0,\infty]$, defined pointwise by

$$Hf(x) = \limsup_{r\to 0} \frac{\max_{|h|=r} |f(x+h) - f(x)|}{\min_{|h|=r} |f(x+h) - f(x)|}. \tag{1.1}$$

We denote the quantities appearing under the limit in (1.1) by

$$L_r f(x) = \max_{|h|=r} |f(x+h) - f(x)|, \qquad \ell_r f(x) = \min_{|h|=r} |f(x+h) - f(x)|,$$

and call them the *stretchings* of f at x over radius r.

1.2 Distortion of a linear mapping. We can see at once that the distortion of a linear map is a constant function. Indeed, if T is a nonsingular linear transformation of the plane, then linearity implies that

$$\frac{\max_{|h|=r} |T(x+h)-T(x)|}{\min_{|h|=r} |T(x+h)-T(x)|} = \frac{\max_{|h|=1} |T(h)|}{\min_{|h|=1} |T(h)|} = \|T\|\,\|T^{-1}\|,$$

regardless of the choice of x and r. We thus may define the *maximal* and the *minimal stretching* of T by

$$L(T) = \max_{|h|=1} |T(h)| \qquad \text{and} \qquad \ell(T) = \min_{|h|=1} |T(h)|. \tag{1.2}$$

We note that T is nonsigular if and only if $\ell(T) > 0$, in which case the distortion $H(T)$ of T is expressed by

$$H(T) = \frac{L(T)}{\ell(T)}. \tag{1.3}$$

By virtue of the Polar Decomposition Theorem, we can also write

$$H(T) = \frac{(L(T))^2}{|\det(T)|} = \frac{|\det(T)|}{(\ell(T))^2}. \tag{1.4}$$

In case if T is singular, we use the convention $H(T) = \infty$.

Geometrically, the distortion of a linear transformation is the ellipticity of the image of the unit disc. Incidentally, there are several ways to do this. Formula (1.3) sees ellipticity as the ratio of the major and minor semiaxes of the ellipse. Quotients in (1.4), on the other hand, compare the area of the ellipse with the areas of its circumscribed and inscribed discs.

Distortion and stretchings of a homeomorphism are closely related to the following notions.

1.3 Definitions. The *maximal derivative* of a mapping f at a point x is the quantity

$$D^{+}f(x) = \limsup_{h\to 0} \frac{|f(x+h)-f(x)|}{|h|}.$$

Similarly, the *minimal derivative* of f at x is given by

$$D^{-}f(x) = \liminf_{h\to 0} \frac{|f(x+h)-f(x)|}{|h|}.$$

Provided that f is a continuous map defined in an open set E,[1] *the functions $D^{+}f$ and $D^{-}f$ are Borel-measurable in E.* To see this, fix a real t and denote $E_t = \{x \in E\colon D^{+}f(x) < t\}$. For every natural k, construct the set

$$E_k = \left\{ x \in E\colon |h| < \frac{1}{k} \text{ implies } x+h \in E \text{ and } \frac{|f(x+h)-f(x)|}{|h|} \leqslant t - \frac{1}{k} \right\}.$$

Continuity of f entails that every E_k is closed in E. Moreover, $E_t = \bigcup_k E_k$, and so E_t is Borel.

1.4 Stretchings of a homeomorphism. Let f be homeomorphic in Ω. At every $x \in \Omega$ and for every $\delta < \operatorname{dist}(x, \partial\Omega)$ we have

$$\sup_{|h|<\delta} \frac{|f(x+h)-f(x)|}{|h|} = \sup_{0<r<\delta} \frac{\max_{|h|=r} |f(x+h)-f(x)|}{r}.$$

This observation, together with an analogous statement for the infimum, leads to the identities

$$D^{+}f(x) = \limsup_{r\to 0} \frac{L_r f(x)}{r} \qquad \text{and} \qquad D^{-}f(x) = \liminf_{r\to 0} \frac{\ell_r f(x)}{r},$$

which hold at every point $x \in \Omega$.

The identities above are somewhat tautological, but they elucidate the connection between the stretchings and maximal/minimal derivatives of a homeomorphism. Moreover, below we use a similar line of reason to obtain a useful inequality which asserts that the distortion function and stretching ℓ_r of a homeomorphism can be used to bound its maximal derivative.

1.5 Theorem. *If f is homeomorphic in Ω, then the inequality*

$$D^{+}f(x) \leqslant Hf(x) \limsup_{r\to 0} \frac{\ell_r f(x)}{r}$$

holds at every point x in Ω.

---

[1] f can be a homeomorphism on a domain, in particular.

The above inequality is a rather elementary consequence of Definition 1.1 of the distortion function. Yet, it will be of great importance later for establishing almost everywhere differentiability of quasiconformal mappings.

Proof Assuming $\delta < \operatorname{dist}(x, \partial\Omega)$, we write

$$\begin{aligned}\sup_{|h|<\delta} \frac{|f(x+h) - f(x)|}{|h|} &= \sup_{r<\delta} \frac{L_r f(x)}{r} \\ &= \sup_{r<\delta} \frac{L_r f(x)}{\ell_r f(x)} \frac{\ell_r f(x)}{r} \\ &\leqslant \sup_{r<\delta} \frac{L_r f(x)}{\ell_r f(x)} \sup_{r<\delta} \frac{\ell_r f(x)}{r}.\end{aligned}$$

Letting $\delta \to 0$ yields the desired inequality. □ Theorem 1.5

From Metric to Analytic Distortion

In Section 1.4 and in Theorem 1.5, we saw how the distortion function of an arbitrary homeomorphism links the homeomorphism's stretchings with its maximal and minimal derivative. Now we explore how all these quantities relate at points of differentiability. The goal is to formulate a local definition of distortion in terms of derivatives.

1.6 Theorem. *Let f be homeomorphic in Ω. If f is differentiable at $x \in \Omega$, then*

$$Hf(x) = H(Df(x)).$$

In essence, this theorem states that two notions of distortion — one given for homeomorphisms by Definition 1.4 and one described by formula (1.3) for linear maps — are consistent, in the sense that the distortion of a differentiable homeomorphism and that of its differential coincide.

Proof By definition of the differential, and in particular by formula (2),

$$|f(x+h) - f(x)| = \big|Df(x)\,h + |h|\,\varepsilon(x,h)\big| \tag{1.5}$$

whenever $|h| < \operatorname{dist}(x, \partial\Omega)$. Fix r such that $0 < r < \operatorname{dist}(x, \partial\Omega)$. Maximizing and minimizing both sides of (1.5) over all h with $|h| = r$ and subsequently dividing by r, we obtain the estimates

$$L(Df(x)) - \max_{|h|=r} |\varepsilon(x,h)| \leqslant \frac{L_r f(x)}{r} \leqslant L(Df(x)) + \max_{|h|=r} |\varepsilon(x,h)|,$$
$$\ell(Df(x)) - \min_{|h|=r} |\varepsilon(x,h)| \leqslant \frac{\ell_r f(x)}{r} \leqslant \ell(Df(x)) + \min_{|h|=r} |\varepsilon(x,h)|.$$

The differential of f is nonsingular at x if and only if $\ell(Df(x)) \neq 0 \neq \ell_r f(x)$. In such scenario the above estimates imply

$$Hf(x) = \limsup_{r\to 0} \frac{L_r f(x)}{\ell_r f(x)} = \frac{L(Df(x))}{\ell(Df(x))} = H(Df(x)).$$

If, on the contrary, $Df(x)$ is singular, then we have $H(Df(x)) = \infty = Hf(x)$, and the proof is complete. □ Theorem 1.6

1.7 Remarks. As a matter of fact, the above proof yields several interrelations between our concepts of interest; the variety of symbols and terminology may cause a good deal of confusion. To sum up, we have

- maximal and minimal derivatives D^+f and D^-f,
- total derivative Df,
- stretchings $L_r f$ and $\ell_r f$,
- stretchings $L(Df)$ and $\ell(Df)$ of the derivative of f,
- infinitesimal distortion function Hf,
- distortion $H(Df)$ of the derivative of f.

In 1.4 and 1.5, we established that assertions

$$D^+f(x) = \limsup \frac{L_r f(x)}{r}, \qquad D^-f(x) = \liminf \frac{\ell_r f(x)}{r},$$
$$D^+f(x) \leqslant Hf(x) \limsup_{r\to 0} \frac{\ell_r f(x)}{r},$$

hold at every point x for every homeomorphism f. Moreover, at points where $Df(x)$ exists, we have

$$D^+f(x) = L(Df(x)) = \max_{|h|=1} |Df(x)\,h| = \|Df(x)\|,$$
$$D^-f(x) = \ell(Df(x)) = \min_{|h|=1} |Df(x)\,h|.$$

If $Df(x)$ exists and is nonsingular, then

$$\max_{|h|=1} |Df(x)\,h| = Hf(x)\,\min_{|h|=1} |Df(x)\,h|,$$

which, in view of (1.4), is equivalent to

$$Hf(x) = \frac{\|Df(x)\|^2}{Jf(x)}. \tag{1.6}$$

Should $Df(x)$ be singular, we set $Hf(x) = H(Df(x)) = \infty$.

Distortion and Rigidity of Conformal Maps

Before proceeding to the geometric distortion, let us recall the definition and some important properties of conformal mappings. Practically every introductory course in complex analysis defines conformal mappings as those that (locally) preserve angles. The following definition is equivalent. Recall that by a rotation we mean a sense-preserving Euclidean isometry.

1.8 Definitions. We say that a linear transformation is *conformal* if it is a positive scalar multiple of a rotation.

A mapping f is *conformal* at a point x if it is differentiable at x and the derivative $Df(x)$ is a conformal linear transformation.

We say that a mapping is *conformal* in Ω if it is conformal at every point of Ω.

The above definition says that a map is conformal if its linearization (cf. page 5) is a similarity transformation.

The Implicit Function Theorem entails that a conformal mapping is locally homeomorphic, since the differential is nonsingular. By the same token, a mapping which is conformal at every point of a simply connected domain is necessarily a homeomorphism on the entire domain.[2]

Being a positive scalar multiple of a rotation, the differential of a conformal map has one repeated eigenvalue. This shows that the first claim of the theorem below is a direct consequence of Definition 1.8. It then immediately follows that conformal maps have unit distortion by virtue of identity (1.6).

[2] Simple connectedness is a necessary requirement.

1.9 Theorem. *If f is conformal at a point $x \in \mathbb{R}^2$, then*

$$\|Df(x)\|^2 = Jf(x),$$

and $Hf(x) = 1$, as a consequence.

All definitions and results we have discussed in this chapter thus far can be extended to dimensions above two with little to no effort. Let us next discuss some exclusively planar phenomena. To begin with, we recall one of the cornerstones of the classical function theory: the Riemann Mapping Theorem.

We say that domains Ω and Ω' are *conformally equivalent* if there exists a conformal bijection Φ such that $\Phi(\Omega) = \Omega'$, that is, if there exists a conformal one-to-one mapping from Ω onto Ω'.

1.10 The Riemann Mapping Theorem. *Every simply connected planar domain other than the plane itself is conformally equivalent to the unit disc.*

We next discuss an example which lead Grötzsch to introduce the idea of quasiconformal maps in late 1920s (although the term 'quasiconformal' was coined by Ahlfors in 1935).

1.11 The problem of Grötzsch. Let Q be a closed square and let Q' be a closed rectangle in the plane. The Riemann Mapping Theorem asserts that there exists a conformal map between the interiors of Q and Q'. Further, Carathéodory's extension theorem states that this mapping extends homeomorphically across the boundary of Q; the extension, however, does not map edges of Q to the corresponding edges of Q' unless the latter is also a square, as we shall verify next.

Note that there is no loss of generality in presuming that $Q = [0,1]\times[0,1]$ and $Q' = [0,L] \times [0,1]$ with L positive, since translations, rotations, and scalings are conformal. Let f be conformal on Q, with the property that $f(Q) = Q'$, and assume that f carries the vertical edges of Q to those of Q'. Fix some height $y \in (0,1)$ and consider a horizontal line segment $I_y = \{t + iy \colon 0 \leqslant t \leqslant 1\}$ in Q. Homeomorphism f sends I_y to some path $\gamma_y = f(I_y)$ which lies in rectangle Q' and connects its vertical sides. We parametrize this path naturally by setting $\gamma_y(t) = f(t + iy)$ for $0 \leqslant t \leqslant 1$. It is clear that the length of γ_y is at least L; with the parametrization of γ_y

in mind, we estimate

$$L \leqslant \int_0^1 |\gamma_y'(t)|\, dt = \int_0^1 \left| \frac{\partial f}{\partial t}(t+iy) \right| dt \leqslant \int_0^1 \left\| Df(t+iy) \right\| \, dt.$$

The Cauchy–Schwarz inequality now yields

$$L^2 \leqslant \int_0^1 \left\| Df(t+iy) \right\|^2 dt,$$

which holds for every y in the interval $(0,1)$. Next, we integrate both sides with respect to the second variable y and apply Fubini's Theorem to obtain

$$L^2 \leqslant \int_0^1 \int_0^1 \left\| Df(t+iy) \right\|^2 dt\, dy = \int_Q \left\| Df(\zeta) \right\|^2 d\zeta. \tag{1.7}$$

We assumed that f is conformal, so $\|Df\|^2 = Jf$ everywhere in Q by Theorem 1.9. The area formula (4) then gives us a bound

$$L^2 \leqslant \int_Q Jf(\zeta)\, d\zeta = |Q'| = L, \tag{1.8}$$

which of course can only be true if $L \leqslant 1$. However, the inverse of f is conformal as well; repeating the same argument for f^{-1} reveals actually that $L \geqslant 1$. We conclude that Q' is necessarily a square.

We just showed that *a conformal mapping can carry vertices of one closed rectangle to vertices of another closed rectangle if and only if the rectangles have the same aspect ratio.* Recognizing this rather simple fact, Grötzsch posed the question: which mappings can transform rectangles in the described fashion and are as close to being conformal as possible? This question was the beginning of the rich theory of quasiconformal mappings.

Much of the discussion in this work is predicated upon the extension of the Riemann Mapping Theorem to doubly connected domains, which we record next. The proof of this topological result can be found in [Kra06].

Let us first say that by an *annulus* we mean a doubly connected domain bounded by two concentric circles. Throughout this document, we use the notation

$$\mathcal{A}(r,R) = \left\{ x \in \mathbb{R}^2 \colon 0 \leqslant r < |x| < R \leqslant \infty \right\}$$

for annuli. We will say that an annulus is *nondegenerate* if both components of its complementary set are continua. Conversely, an annulus is *degenerate* if $r = 0$ or $R = \infty$.

1.12 The Riemann Mapping Theorem for doubly connected domains. *Every doubly connected planar domain is conformally equivalent to an annulus.*

By the Riemann Mapping Theorem 1.10, all simply connected strict subdomains of the plane are conformally equivalent. But is the same true of doubly connected domains? The proposition below gives the negative answer.

1.13 Proposition (Conformal Equivalence of Annuli). *Annuli $\mathcal{A}(r_1, R_1)$ and $\mathcal{A}(r_2, R_2)$ are conformally equivalent if and only if $R_1/r_1 = R_2/r_2$.*

This assertion is another illustration for the lack of flexibility in the class of conformal mappings, similar in spirit to the discussion in § 1.11. We should only mention that this proposition can be easily deduced from some of the results on conformal invariance which we discuss in the sequel; namely, it follows from Theorem 1.19 and computations in § 1.22. A complete, albeit somewhat technical proof using different techniques can be found in [Rud87, Theorem 14.22].

Ring Capacity

Thus far, we discussed ways to define the distortion of a mapping locally, in infinitesimal terms. An entirely different, geometric approach allows us to gauge the distortion globally. The idea is to take some conformal invariant (i.e. a quantity which remains unchanged under a conformal transformation) and measure how much a given mapping distorts this invariant.

Note that all of the following definitions and results readily generalize to higher dimensions.

1.14 An interim remark. In this text, we use a conformal invariant called *ring capacity.* While capacity affords a neat physical interpretation (see Remark 1.17), its connection to the foregoing infinitesimal notions of distortion is obscure. It would be instructive — at this point in our discussion — to first turn to another important conformal invariant, called the *extremal length* (EL). The reason is that the definition of EL is, in a way, a global formulation of the analytic distortion. For coherency of the text we choose to discuss EL separately in Appendix A (page 93).

1.15 Ring domains. Let us begin by saying that a *ring* on the Riemann sphere $\overline{\mathbb{C}}$ is a domain whose complement in $\overline{\mathbb{C}}$ has precisely two components.[3] We say that a ring is *nondegenerate* if both components of the complementary set are continua, and *degenerate* otherwise.

Given a ring on $\overline{\mathbb{C}}$, we can always use a suitable Möbius transformation to obtain a ring whose complement contains the point ∞. Since Möbius transformations are conformal, such normalization preserves each and every conformal invariant assotiated with the ring. After mapping such ring from the sphere to the finite complex plane through a stereographic projection, we come to the following definition of a ring domain in the plane.

A *ring* in $\mathbb{C}$ is a planar domain whose complement has precisely two components, one of which is compact and the other closed and unbounded. The symbol $\mathcal{R}(E, F)$ will signify a ring with the complementary components E and F:

$$\mathcal{R}(E, F) = \mathbb{R}^2 \setminus \{E \cup F\}.$$

Where it is important to discern the components of the complementary set, we shall denote the bounded component by C_1 and the ubounded component by C_0 (unless we state otherwise).

Every planar ring is doubly connected.[4] We say that a ring *separates* two sets, if these sets lie in different complementary components of the ring. We transfer the notion of degeneracy from the Riemann sphere: a ring $\mathcal{R}(C_0, C_1)$ is nondegenerate if both C_0 and C_1 are continua, and degenerate if C_1 consists of a finite point or if $C_0 = \{\infty\}$.

1.16 Definitions. With every ring $R = \mathcal{R}(C_0, C_1)$ we associate the class of *admissible functions* $\mathrm{Adm}(R)$ by requiring that each member $u \in \mathrm{Adm}(R)$ is continuous in R up to the boundary (in the topology of $\overline{\mathbb{C}}$), enjoys the ACL property in R, and satisfies the boundary conditions

$$u\big|_{C_0} \equiv 0 \quad \text{and} \quad u\big|_{C_1} \equiv 1.$$

[3] Recall that the Riemann sphere is endowed with the 'cap' topology, whose base consists of the discs $\mathbb{D}_r(x)$ around every point $x \in \mathbb{C}$ with $0 < r < \infty$ and the caps $\mathbb{D}_r(\infty) = \{\infty\} \cup \{x \in \mathbb{C} \colon |x| > r^{-1}\}$.

[4] In higher dimensions, a ring domain need not be doubly connected, nor a doubly connected domain need be a ring: think of $\mathbb{R}^3$ minus two disjoint balls and $\mathbb{R}^3$ minus a torus.

Because an ACL function has a formal gradient at almost every point, the integral

$$\int_B |\nabla u(x)|^2 \, dx,$$

called the *Dirichlet energy* of u in B, makes sense for every admissible u and for every Borel set B. We thus define the *capacity* of a ring R as the extended real number

$$\mathrm{Cap}(R) = \inf_{u \in \mathrm{Adm}(R)} \int_{\mathbb{R}^2} |\nabla u(x)|^2 \, dx.$$

We shall also make use of the quantity

$$\mathrm{Mod}(R) = \frac{2\pi}{\mathrm{Cap}(R)},$$

which we call the *module* of a ring R.

1.17 Remark. A reader familiar with electromagnetism will find that ring capacity has an obvious physical interpretation: capacitance. Often in the literature, instead of defining the ring $R = \mathcal{R}(C_0, C_1)$, authors work with the pair $C = (C_0, C_1)$ of sets, called the *condenser*. The components C_0 and C_1 are called the *plates* of condenser C, and the ring R itself is referred to as the *field* of C.

We record in the following lemma that the infimum in the definition of $\mathrm{Cap}(R)$ is actually attained by some smooth function satisfying even more restrictive boundary conditions.

1.18 Lemma. *The capacity of a ring R remains unchanged if we restrict our choice of admissible functions to any of the following classes:*

(*i*) $\mathrm{Adm}_0(R) = \{u \in \mathrm{Adm}(R) \colon u$ *is compactly supported in* $R \cup C_1\}$,

(*ii*) $\mathrm{Adm}_1(R) = \{u \in \mathrm{Adm}_0(R) \colon u \equiv 1$ *in a neighbourhood of* $C_1\}$, *or*

(*iii*) $\mathrm{Adm}_\infty(R) = \mathrm{Adm}_1(R) \cap C^\infty(\mathbb{C})$.

PROOF We can see why this is true by a standard mollification argument. Due to the inclusion $\mathrm{Adm}_1(R) \subset \mathrm{Adm}_0(R)$, part (*i*) of the claim will automatically follow once we prove part (*ii*). Let $u \in \mathrm{Adm}(R)$, fix $\epsilon \in (0, \frac{1}{2})$, and define a piecewise linear cutoff function ϕ_ϵ on $\mathbb{R}$ by

$$\phi_\epsilon(t) = \begin{cases} 0, & t < \epsilon, \\ \dfrac{t-\epsilon}{1-2\epsilon}, & \epsilon \leqslant t < 1-\epsilon, \\ 1, & 1-\epsilon \leqslant t. \end{cases}$$

Observe that ϕ_ϵ is Lipschitz continuous, whence the composition $u_\epsilon = \phi_\epsilon \circ u$ is ACL. Moreover, u_ϵ satisfies $|\nabla u_\epsilon| \leqslant (1-2\epsilon)^{-1}|\nabla u|$ at points where the gradient of u exists — almost everywhere. Further, the support of u_ϵ lies in the set $\{x\colon u(x) \geqslant \epsilon\}$, and $u_\epsilon \equiv 1$ everywhere in $\{x\colon u(x) > 1-\epsilon\}$. From this we deduce that u_ϵ is of class $\mathrm{Adm}_1(R)$. Because $\mathrm{Adm}_1(R) \subset \mathrm{Adm}(R)$, we have

$$\begin{aligned} \mathrm{Cap}(R) &\leqslant \inf_{v \in \mathrm{Adm}_1(R)} \int_R |\nabla v(x)|^2\, dx \\ &\leqslant \liminf_{\epsilon\to 0} \int_R |\nabla u_\epsilon(x)|^2\, dx \\ &\leqslant \liminf_{\epsilon\to 0} (1-2\epsilon)^{-1} \int_R |\nabla u(x)|^2\, dx \\ &= \int_R |\nabla u(x)|^2\, dx. \end{aligned}$$

Admissible function u was selected at random, so we can take infimum of the right side, and the claim follows.

Suppose now that $v \in \mathrm{Adm}_1(R)$. We can define a sequence $v_\epsilon = \varphi_\epsilon * v$ by convoluting v with a sequence of standard mollifiers φ_ϵ (cf. page 9). For sufficiently small ϵ, v_ϵ lies in $\mathrm{Adm}_\infty(R)$. Thus

$$\begin{aligned} \mathrm{Cap}(R) &\leqslant \inf_{w \in \mathrm{Adm}_\infty(R)} \int_R |\nabla w(x)|^2\, dx \\ &\leqslant \lim_{\epsilon\to 0} \int_R |\nabla v_\epsilon(x)|^2\, dx \\ &= \int_R |\nabla v(x)|^2\, dx, \end{aligned}$$

because convolutions ∇v_ϵ converge to ∇v in L^2–norm. As before, we take the infimum of the right side over $v \in \mathrm{Adm}_1(R)$, finishing the proof. □ Lemma 1.18

1.19 Theorem. *Ring capacity and ring module are conformally invariant.*

Proof Let u be an admissible function on R; by Lemma 1.18 we may assume that $u \in C^\infty(R)$. Let f be a conformal homeomorphism of R. The assertion of the theorem rests on a fundamental fact that the Dirichlet energy of u is conformally invariant. To prove this, we first note that

$$\nabla(u \circ f)(x) = D^t f(x)\,(\nabla u \circ f)(x)$$

holds at every point $x \in R$ by the chain rule. Conformality of f entails that the differential of f is a multiple of an orthogonal transformation, with the scaling factor equal to the square root of the Jacobian determinant. Thus the gradient norm of $u \circ f$ satisfies

$$|\nabla(u \circ f)(x)|^2 = Jf(x)\,|(\nabla u \circ f)(x)|^2,$$

and hence, by the change of variables formula (5),

$$\int_R |\nabla(u \circ f)(x)|^2\,dx = \int_R Jf(x)\,|(\nabla u \circ f)(x)|^2\,dx = \int_{f(R)} |\nabla u(y)|^2\,dy,$$

where we set $y = f(x)$.

Finally, we note that $u \circ f$ is a smooth admissible function on $f(R)$ (in particular, it has the required boundary values). Now the claim follows because u was chosen at random. □ THEOREM 1.19

1.20 Theorem. *Ring capacity has the following properties.*

(*i*) *Capacity is monotone decreasing with respect to inclusions. That is, if R_1, R_2 are rings and if $R_1 \subset R_2$, then* $\operatorname{Cap} R_1 \geqslant \operatorname{Cap} R_2$.

(*ii*) *If rings $R_1, R_2, \ldots$ exhaust*[5] *the ring R, then*

$$\lim_{n\to\infty} R_n = \operatorname{Cap} R.$$

PROOF The first assertion follows immediately from the definition of ring capacity.

To prove the second assertion, note that (i) entails $\operatorname{Cap} R_n \geqslant \operatorname{Cap} R_{n+1}$ for every k, so the sequence $\operatorname{Cap} R_n$ is a decreasing sequence bounded below by $\operatorname{Cap} R$, and hence has a limit. Assume towards a contradiction that $\lim_{k\to\infty} \operatorname{Cap} R_n > \operatorname{Cap} R$. Then there exists a function v of the class

[5]We say that sets $A_1, A_2, \ldots$ form an *exhausting sequence* for set A, if $\overline{A_n} \subset A_{n+1}$ for all natural n, and $A = \bigcup_n A_n$.

$\mathrm{Adm}(R)$, whose Dirichlet integral satisfies

$$\lim_{n\to\infty} \operatorname{Cap} R_n > \int_R |\nabla v(x)|^2\, dx. \tag{1.9}$$

For natural $k > 2$, the sets $\{x \in R\colon v(x) < 1/k\}$ and $\{x \in R\colon v(x) > 1 - 1/k\}$ are open. Thus the exhausting sequence of R has a ring $R_{n_k} = \mathcal{R}(C_{n_k 0}, C_{n_k 1})$ such that $v < 1/k$ on $\partial C_{n_k 0}$ and $v > 1 - 1/k$ on $\partial C_{n_k 1}$. This implies that the function

$$w = \max\left(\min\left(\frac{k\,v - 1}{k-2}, 1\right), 0\right)$$

is admissible for R_{n_k}. We therefore have

$$\begin{aligned}\operatorname{Cap} R_{n_k} &\leqslant \int_{R_{n_k}} |\nabla w(x)|^2\, dx \\ &\leqslant \int_R \left|\nabla\left(\frac{k\,v(x) - 1}{k-2}\right)\right|^2 dx = \left(\frac{k}{k-2}\right)^2 \int_R |\nabla v(x)|^2\, dx,\end{aligned}$$

which contradicts (1.9) for large k. This confirms the second assertion. □ THEOREM 1.20

1.21 Remark. The following useful variation of assertion (*ii*) above holds.

If $R_1, R_2, \dots$ is a sequence of rings whose boundary components converge uniformly (in the set distance) to the boundary components of the ring R, then

$$\lim_{n\to\infty} R_n = \operatorname{Cap} R.$$

This result, together with the geometric definition $\mathrm{QC}_{\mathrm{G}'}$ of Remark 2.3 below, can be used to show in a very straightforward manner that a homeomorphic limit of a compactly convergent sequence of K–quasiconformal maps is K–quasiconformal (Theorem 5.8).

1.22 Capacity of an annulus. Let us compute the capacity of an annulus — to illustrate the concept as well as to obtain a useful estimate to be used in the sequel. Assume $0 < r < R < \infty$, and observe that the annulus $\mathcal{A}(r, R)$ is conformally equivalent to the 'normalized' annulus $A = \mathcal{A}(1, R/r)$.

Let u be admissible for A. Fix an angle θ and let γ_θ be the radial path connecting the components of ∂A; we parametrize it by $\gamma_\theta(t) = te^{i\theta}$. As $u \in \mathrm{Adm}(A)$ varies between 0 and 1 in A, every path which connects the components of ∂A must have $|\nabla u|$–length[6] at least 1, and hence

$$1 \leqslant \int_1^{R/r} \left|\nabla u\big(te^{i\theta}\big)\right| dt.$$

Integrating both sides with respect to argument θ and applying the Cauchy–Schwarz inequality gives

$$\begin{aligned} 2\pi &\leqslant \int_0^{2\pi}\int_1^{R/r} \left|\nabla u\big(te^{i\theta}\big)\right| dt\, d\theta \\ &\leqslant \left(\int_0^{2\pi}\int_1^{R/r} \left|\nabla u\big(te^{i\theta}\big)\right|^2 t\, dt\, d\theta\right)^{\frac{1}{2}} \left(\int_0^{2\pi}\int_1^{R/r} \frac{1}{t}\, dt\, d\theta\right)^{\frac{1}{2}}, \end{aligned}$$

whence

$$2\pi \left(\ln \frac{R}{r}\right)^{-1} \leqslant \int_{\mathbb{R}^2} |\nabla u(x)|^2\, dx$$

by the Fubini's Theorem. Because $u \in \mathrm{Adm}(A)$ was selected at random, the quantity on the left is a lower bound for $\mathrm{Cap}(A)$. At the same time, the function

$$v(x) = \begin{cases} 1, & |x| \leqslant 1 \\ \dfrac{\ln R - \ln |x|}{\ln R - \ln r}, & 1 < |x| < R/r \\ 0, & |x| \geqslant R/r \end{cases}$$

is an admissible density in A; its gradient norm is given by

$$|\nabla v(x)| = \frac{1}{|x|\, \ln \frac{R}{r}}$$

in the interior of A and zero elsewhere. The Dirichlet energy of v is

$$\int_{\mathbb{R}^2} |\nabla v(x)|^2\, dx = \left(\ln \frac{R}{r}\right)^{-2} \int_0^{2\pi}\int_r^R \frac{1}{r^2}\, r\, dr\, d\theta = 2\pi \left(\ln \frac{R}{r}\right)^{-1},$$

and it surely dominates $\mathrm{Cap}(A)$. Combining these observations, we arrive

[6]If $\rho \geqslant 0$ is a Borel function on $\mathbb{R}^2$ and γ is a path, we say that the quantity $\int_\gamma \rho(s)\, |ds|$ is the ρ–length of γ.

at the formulas

$$\mathrm{Cap}(A) = 2\pi \left(\ln \frac{R}{r} \right)^{-1}, \tag{1.10}$$

$$\mathrm{Mod}(A) = \ln \frac{R}{r}. \tag{1.11}$$

Here is a good place to remark that the factor 2π in the definition of the ring module is entirely optional; its presence is justified by the simplicity of expression (1.11).[7]

Further, observe that formula (1.10) and conformal invariance of capacity entail Proposition 1.13.

On a final note, the Riemann Mapping Theorem for doubly connected domains (Theorem 1.12), together with Proposition 1.13, means that every annulus $\mathcal{A}(1, R/r)$ gives rise to an equivalence class — modulo a conformal mapping — of ring domains, and formulæ (1.10) and (1.11) are valid if we replace the annulus A with any conformal image thereof.

Defining Distortion Globally

We are now at the position where we can give a geometric definition of distortion in terms of ring capacity.

1.23 Definitions. Let f be a homeomorphism on a planar domain Ω. We define the *inner ring distortion* $K_I(f)$ and *outer ring distortion* $K_O(f)$ of f by

$$K_I(f) = \inf_{K \in (0,\infty)} \left\{ K\colon \ \mathrm{Cap}(f(R)) \leqslant K \ \mathrm{Cap}(R) \text{ for every ring } R \subset\subset \Omega \right\},$$

$$K_O(f) = \inf_{K \in (0,\infty)} \left\{ K\colon \ \mathrm{Cap}(R) \leqslant K \ \mathrm{Cap}(f(R)) \text{ for every ring } R \subset\subset \Omega \right\}.$$

The *maximal ring distortion*, or just the *ring distortion* $K(f)$ is defined by

$$K(f) = \max \left\{ K_I(f),\, K_O(f) \right\}.$$

We adopt the convention that K_I or K_O equals ∞ if the corresponding set over which the infimum is taken is empty.

The ring distortion of f in Ω can of course be written as

$$K(f) = \inf_{K \in (0,\infty)} \left\{ K\colon \frac{1}{K} \leqslant \frac{\mathrm{Cap}(f(R))}{\mathrm{Cap}(R)} \leqslant K \text{ for every ring } R \subset\subset \Omega \right\}.$$

[7] Some authors also normalize the capacity by the factor $(2\pi)^{-1}$.

It is clear that replacing ring capacity with ring module yields the same number $K(f)$:

$$K(f) = \inf_{K\in(0,\infty)} \left\{ K\colon \frac{1}{K} \leqslant \frac{\mathrm{Mod}(f(R))}{\mathrm{Mod}(R)} \leqslant K \text{ for every ring } R \subset\subset \Omega \right\}.$$

We can observe the following immediate properties of these geometric distortion functions.

1.24 Lemma. *Let $f\colon \Omega \to \Omega'$ and $g\colon \Omega' \to \Omega''$ be homeomorphisms. The following properties hold:*

(*i*) $K_I(f^{-1}) = K_O(f), \quad K_O(f^{-1}) = K_I(f), \quad K(f^{-1}) = K(f);$

(*ii*) $K_I(g\circ f) \leqslant K_I(g)\,K_I(f), \quad K_O(g\circ f) \leqslant K_O(g)\,K_O(f),$
$K(g\circ f) \leqslant K(g)\,K(f).$

PROOF Since $f\colon \Omega \to \Omega'$ is homeomorphic, it follows at once that for every ring $R \subset\subset \Omega$ and for $K \in (0,\infty)$, the condition

$$\mathrm{Cap}(f(R)) \leqslant K\ \mathrm{Cap}(R)$$

is equivalent to

$$\mathrm{Cap}(S) \leqslant K\ \mathrm{Cap}(f^{-1}(S)),$$

where $S = f(R) \subset\subset \Omega'$. We thus see that

$$K_I(f) = K_O(f^{-1}) \quad \text{and} \quad K_O(f) = K_I(f^{-1}).$$

The last equality in (*i*) follows from the definition.

Inequalities in (*ii*) are almost as trivial. The first inequality holds trivially if either $K_I(g) = \infty$ or $K_I(f) = \infty$. Otherwise, both of these distortions are finite, and for every ring R we have

$$\mathrm{Cap}(g(f(R))) \leqslant K_I(g)\ \mathrm{Cap}(f(R)) \leqslant K_I(g)\,K_I(f)\ \mathrm{Cap}(R).$$

The rest is proved in a similar manner. For the details, see for example [GMP17]. □ LEMMA 1.24

Chapter 2

Definitions of Quasiconformality

In light of the foregoing discussion, we define quasiconformality as follows.

2.1 Definitions. Let f be a sense-preserving homeomorphism defined in a planar domain Ω.

$\mathbf{QC_M}$ We say that f is (*metrically*) *quasiconformal* in Ω if its linear distortion is uniformly bounded. Specifically, f is *H–quasiconformal* if there is a constant H such that

$$Hf(x) = \limsup_{r \to 0} \frac{\max_{|h|=r} |f(x+h) - f(x)|}{\min_{|h|=r} |f(x+h) - f(x)|} \leqslant H < \infty$$

at every point x in Ω.

$\mathbf{QC_A}$ We say that f is (*analytically*) *K–quasiconformal* in Ω, if it belongs to the Sobolev class $W^{1,2}_{\text{loc}}(\Omega)$, is differentiable almost everywhere in Ω, and there is a constant $K < \infty$ such that the distortion inequality

$$\|Df(x)\|^2 \leqslant K\, Jf(x)$$

holds at almost every x in Ω.

$\mathbf{QC_G}$ We say that f is (*geometrically*) *K–quasiconformal* in Ω if the outer ring distortion $K_O(f)$ is finite, or, equivalently, if

$$\text{Cap}(R) \leqslant K \ \text{Cap}(f(R))$$

for some nonzero finite constant K and for every ring $R \subset\subset \Omega$.

The main objective of this chapter is to prove the following result.

2.2 Theorem. *Definitions* $\mathrm{QC_M}$*,* $\mathrm{QC_A}$*, and* $\mathrm{QC_G}$ *are equivalent.*

We recall from the previous chapter that if $f\colon \Omega \to \mathbb{R}^2$ is conformal in Ω, then the identities

$$Hf(x) = 1, \qquad \|Df(x)\|^2 = Jf(x), \qquad \mathrm{Cap}(f(R)) = \mathrm{Cap}(R)$$

hold at every point $x \in \Omega$ and for every ring $R \subset\subset \Omega$. In other words, *every conformal map is 1–quasiconformal.* The converse is also true: *every 1–quasiconformal mapping is conformal.* This is the assertion of Weyl's Lemma 4.12, which we prove later.

Each of the three Definitions 2.1 states that QC maps are 'almost' conformal homeomorphisms, which is in accordance with the name. Definitions $\mathrm{QC_M}$ and $\mathrm{QC_A}$ capture this idea locally. Both assert that QC maps transform infinitesimal circles to ellipses whose eccentricities are uniformly bounded. The geometric definition is a global formulation of the same idea. It states that $\mathrm{QC_G}$ mappings distort capacities of rings in a restricted manner.[1]

We warn the reader that, in general, distortion bounds in $\mathrm{QC_A}$ and $\mathrm{QC_G}$ need not be the same for a quasiconformal map.

2.3 Remark on the geometric definition. In Chapter 5, we shall prove that the inverse of every quasiconformal map is quasiconformal. In view of the property $K_O(f) = K_I(f^{-1})$ of Lemma 1.24, this entails that for every homeomorphism f, $K_O(f)$ is finite if and only if $K_I(f)$ is.[2] Hence, Definition 2.1–$\mathrm{QC_G}$ admits an equivalent — and most widespread — formulation as follows.

$\mathrm{QC_{G'}}$ A homeomorphism f is *geometrically quasiconformal* in Ω if and only if the ring distortion $K(f)$ is finite. In particular, f is (geometrically) K–quasiconformal in Ω if there is a finite constant K such that for every ring $R \subset\subset \Omega$ we have

$$\frac{1}{K}\ \mathrm{Cap}(R) \leqslant \mathrm{Cap}(f(R)) \leqslant K\ \mathrm{Cap}(R).$$

Throughout this chapter though, we stick with the 'orignal' definition $\mathrm{QC_G}$.

[1] A geometric definition based on *extremal length* elucidates the connection between the infinitesimal and global formulations; see Remark 1.14 and Appendix A.

[2] An even stronger result holds in two dimensions: the outer and inner ring distortions $K_O(f)$ and $K_I(f)$ coincide if f is a homeomorphism between planar domains.

Metric Implies Analytic

We begin the cycle of implications by demonstrating that a homeomorphism of bounded infinitesimal distortion is analytically quasiconformal.

2.4 Theorem ($\mathrm{QC_M} \Rightarrow \mathrm{QC_A}$). *Let f be homeomorphic on Ω. If $Hf \leqslant H < \infty$ everywhere in Ω, then $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$, f is differentiable almost everywhere in Ω, and*

$$\|Df(x)\|^2 \leqslant H\, Jf(x) \tag{2.1}$$

at almost every $x \in \Omega$.

We first establish differentiability almost everywhere with the help of the following well-known result.

2.5 The Rademacher–Stepanov Theorem. *A measurable mapping f is differentiable almost everywhere in the set*

$$\left\{x \in \mathbb{R}^2 \colon D^+f(x) < \infty\right\}.$$

Recall that the classical Rademacher Theorem asserts that Lipschitz functions are differentiable almost everywhere, see for instance [EG15]. It generalizes to the Rademacher–Stepanov Theorem; an accessible proof can be found in [Mal99].

2.6 Lemma. *A homeomorphism f defined in Ω is differentiable at almost every point of the set*

$$E = \left\{x \in \Omega \colon Hf(x) < \infty\right\}.$$

PROOF In light of the Rademacher–Stepanov Theorem 2.5 it suffices to show that $D^+f(x)$ is finite at almost every point $x \in E$. Theorem 1.5 asserts that

$$D^+f(x) \leqslant Hf(x) \limsup_{r\to 0} \frac{\ell_r f(x)}{r} \tag{2.2}$$

at every x in Ω. Therefore, it suffices to show that $\limsup_{r\to 0} \frac{\ell_r f(x)}{r}$ is finite almost everywhere in E. Fix $x \in E$ and suppose $0 < r < \mathrm{dist}(x, \partial\Omega)$. Since f is homeomorphic, the disc of radius $\ell_r f(x)$ centred at $f(x)$ is contained in the image $f(\mathbb{D}_r(x))$, whence

$$\frac{(\ell_r f(x))^2}{r^2} \leqslant \frac{|f(\mathbb{D}_r(x))|}{|\mathbb{D}_r(x)|}.$$

It follows that

$$|D^+ f(x)|^2 \leqslant (Hf(x))^2 \, D^+ \nu_f(x), \tag{2.3}$$

where ν_f is the pullback of the Lebesgue measure by f. Again, f is a homeomorphism, so ν_f is a Borel measure on Ω, and its maximal derivative $D^+\nu_f$ is finite almost everywhere in Ω by the Radon–Nikodym Theorem (cf. pp. 6–7). The hypothesis of the Rademacher–Stepanov Theorem is, therefore, satisfied almost everywhere in E, and we are done. □ LEMMA 2.6

2.7 Corollary. *If f satisfies the hypothesis of Theorem 2.4, then it is differentiable almost everywhere in Ω and* (2.1) *holds at almost every $x \in \Omega$.*

PROOF Differentiability almost everywhere follows from Lemma 2.6 above. In view of remarks made in § 1.7, the distortion bound (2.1) now comes for free. Indeed, if x is a point of differentiability of f, then

$$|D^+ f(x)| = \max_{|h|=1} |Df(x)h|$$

and

$$\limsup_{r\to 0} \frac{\ell_r f(x)}{r} = \lim_{r\to 0} \frac{\ell_r f(x)}{r} = \min_{|h|=1} |Df(x)h|.$$

Hence (2.2) implies

$$\max_{|h|=1} |Df(x)h| \leqslant Hf(x) \min_{|h|=1} |Df(x)h|,$$

and consequently

$$\|Df(x)\|^2 \leqslant Hf(x)\, Jf(x).$$

This finishes the proof. □ COROLLARY 2.7

The proof of Lemma 2.6 yields another remarkable conclusion: the derivative of a metrically quasiconformal map is locally square-integrable.

2.8 Corollary. *If f is $\mathrm{QC_M}$ in Ω, then $Df \in L^2_{\mathrm{loc}}(\Omega)$.*

PROOF The generalized Jacobian of f coincides with the pointwise Jacobian Jf at points of differentiability (almost everywhere) and is integrable by the Radon–Nikodym Theorem. Hence, for every compact subset $F \subset \Omega$, we have

$$\int_F \|Df(x)\|^2 \, dx \leqslant H \int_F |Jf(x)| \, dx \leqslant H \, |f(F)| < \infty$$

by virtue of (2.1). This concludes the proof. □ COROLLARY 2.8

To complete the proof of Theorem 2.4 it remains to show that metrically quasiconformal maps possess sufficient Sobolev regularity. On the strength of the ACL characterization of the class $W^{1,2}_{\mathrm{loc}}$ (cf. p. 8) and Corollary 2.8, we only need to demonstrate that $\mathrm{QC_M}$ homeomorphisms enjoy the ACL property. Below we present an adaptation of the n–dimensional version of this from Jussi Väisälä's lecture notes, [Väi06, Theorem 31.2].

We will use the following elementary lemma.

2.9 Lemma. *Let γ be a path which carries the interval $I = [a, b]$ injectively to $\mathbb{R}^2$. Suppose that for every $\epsilon > 0$ there is $\delta > 0$ such that, whenever $\{\Delta_i\}_{i=1}^k$ is a finite family of disjoint closed subintervals of I with $\sum_{i=1}^k \mathcal{H}^1(\Delta_i) < \delta$, then $\mathcal{H}^1\big(\bigcup_{i=1}^k \gamma(\Delta_i)\big) < \epsilon$. Under this hypothesis, γ is absolutely continuous.*

PROOF Let a_i and b_i denote the endpoints of the subinterval Δ_i as defined in the condition of the lemma. We use the properties of the Hausdorff measure (see for instance [EG15, Section 2.2]) to obtain the bound

$$\begin{aligned}\sum_{i=1}^k |\gamma(a_i) - \gamma(b_i)| \leqslant \sum_{i=1}^k \operatorname{diam}(\gamma(\Delta_i)) \\ \leqslant \sum_{i=1}^k \mathcal{H}^1(\gamma(\Delta_i)) \leqslant \mathcal{H}^1\left(\bigcup_{i=1}^k \gamma(\Delta_i)\right) < \epsilon.\end{aligned}$$

□ LEMMA 2.9

2.10 Lemma. *Every homeomorphism of bounded metric distortion is ACL.*

PROOF Let f be homeomorphic on Ω, and suppose that $Hf < H < \infty$ everywhere in Ω. Consider a closed rectangle $Q = [x_1, x_2] \times [y_1, y_2]$ contained in Ω. For a Borel subset A of (x_1, x_2), we denote $E_A = A \times (y_1, y_2)$. Since the orthogonal projection onto the horizontal axis is a measurable mapping, set E_A is Borel. Homeomorphism f has a continuous (hence measurable) inverse, so $f(E_A)$ is a Borel set, too. We thus may define a measure ν on the interval (x_1, x_2) by setting $\nu(A) = |f(E_A)|$. By the Lebesgue's theorem, density $D\nu$ exists and is finite at almost every point x in (x_1, x_2). Let us fix such point x; we shall prove that f is absolutely continuous on the open line segment $E_x = \{x\} \times (y_1, y_2)$.

Because f is homeomorphic, its restriction $f|_{E_x}$ to the vertical line segment defines an injective path in the plane. Now let F be a compact subset of E_x. We wish to find an upper bound for $\mathcal{H}^1(f(F))$ in terms of $\mathcal{H}^1(F)$, for then Lemma 2.9 would ensure that f is AC on E_x.

STEP 1 We first exhaust F by compacta on which we can gain control over the maximal stretching in terms of the minimal stretching. We assumed $Hf(x) < H$ at every x in Ω; the definition of Hf means that for sufficiently large natural k — large enough so that $1/k < \operatorname{dist}(F, \partial Q)$ — we can construct nonempty sets

$$F_k = \left\{x \in F\colon 0 < r < 1/k \text{ implies } L_r f(x) \leqslant H\, \ell_r f(x)\right\}.$$

Clearly, $F_k \subset F_{k+1}$ and $\bigcup F_k = F$. Moreover, by continuity of f, each F_k is compact. This implies $\mathcal{H}^1(f(F)) = \lim_{k\to\infty} \mathcal{H}^1(f(F_k))$ and so it is enough to find a bound for the measure of F_k for some fixed k.

For the next step, we will need an auxiliary covering lemma whose proof we postpone until after the current proof.

2.11 Lemma. *Suppose that F is compact subset of the real line and let $\epsilon > 0$ be fixed. There exists $\delta > 0$ with the following property: for every radius $r \in (0, \delta)$ one can find finitely many points $z_1, \dots, z_m$ in F such that*

(*i*) *the intervals $(z_i - r, z_i + r)$ cover F;*

(*ii*) *every point of F belongs to at most two subintervals $(z_i - r, z_i + r)$, and*

(*iii*) *a bound $mr < \mathcal{H}^1(F) + \epsilon$ holds.*

STEP 2 In the previous step, we chose k; now fix an $\epsilon > 0$ and a Hausdorff measure parameter $s > 0$. We apply Lemma 2.11 to set F_k and ϵ — this yields a number δ. Homeomorphism f is uniformly continuous in Q, so we can select $r \in (0, \min\{\delta,\, k^{-1}\})$ such that $\operatorname{diam}(f(\mathbb{D}_r(z))) < s$ for every

$z \in Q$.

Now let $z_1, \ldots, z_m$ be the points of F_k provided by Lemma 2.11 for our choice of r. The n the discs $D_i = \mathbb{D}_r(z_i)$ cover F_k, whence

$$\mathcal{H}^1_s(f(F_k)) \leqslant \sum_{i=1}^{m} \operatorname{diam}(f(D_i)) \leqslant 2 \sum_{i=1}^{m} L_r f(z_i).$$

Squaring both sides and applying the Cauchy–Schwarz inequality for sums yields

$$\Big(\mathcal{H}^1_s(f(F_k))\Big)^2 \leqslant 4m \sum_{i=1}^{m} \big(L_r f(z_i)\big)^2.$$

Construction of set F_k (essentially the fact that f is of bounded distortion) allows us to estimate

$$\sum_{i=1}^{m} \big(L_r f(z_i)\big)^2 \leqslant H^2 \sum_{i=1}^{m} \big(\ell_r f(z_i)\big)^2 \leqslant \frac{H^2}{\pi} \sum_{i=1}^{m} |f(D_i)|.$$

Since $r < k^{-1} < \operatorname{dist}(F, \partial Q)$, all discs D_i are contained in the strip $E_D = D \times (y_1, y_2)$, where $D = (x - r, x + r)$ is the projection of the discs onto the x–axis. Furthermore, at most two discs overlap at any given point of F_k by property (ii) of Lemma 2.11. This means that

$$\sum_{i=1}^{m} |f(D_i)| \leqslant 2|f(E_D)| = 2\nu((x - r, x + r)).$$

Collecting the above inequalities and recalling property (iii) of Lemma 2.11 and that $F_k \subset F$, we thus obtain the bound

$$\Big(\mathcal{H}^1_s(f(F_k))\Big)^2 \leqslant \frac{32H^2}{\pi} \, \frac{\nu((x - r, x + r))}{2r} \, (\mathcal{H}^1(F) + \epsilon).$$

STEP 3 As the final step, we let first r, then ϵ, and then s tend to 0, which yields an estimate

$$\Big(\mathcal{H}^1(f(F_k))\Big)^2 \leqslant \frac{32H^2}{\pi} \, D\nu(x) \, \mathcal{H}^1(F).$$

Since $\mathcal{H}^1(f(F_k)) \to \mathcal{H}^1(f(F))$ as $k \to \infty$ and $D\nu(x) < \infty$, this finishes the proof. □ LEMMA 2.10

PROOF OF LEMMA 2.11 Choose an open subset G of the real line, such that $F \subset G$ and $\mathcal{H}^1(G) < \mathcal{H}^1(F) + \epsilon$. We claim that $\delta = \operatorname{dist}(F, \partial G)$ has the desired property. Assume $0 < r < \delta$. The collection of intervals $\{(z - r, z + r)\}_{z \in F}$ covers F, and, by compactness, we can extract a finite subcover of F consisting of intervals $I_i = (z_i - r, z_i + r)$ with $z_i < z_{i+1}$. For $i = 1, 2, \ldots$, if I_i meets I_{i+2}, we discard I_{i+1} and obtain a family which still satisfies the property (i). Proceeding in this manner, we obtain in finitely many steps a cover $I_1, \ldots, I_m$ that satisfies properties (i) and (ii). Finally, it is not difficult to verify that properties (i) and (ii) imply (iii). Indeed, let χ_i denote the indicator function of I_i for $i = 1, \ldots, m$. We then have

$$\begin{aligned} 2mr = \sum_{i=1}^{m} \mathcal{H}^1(I_i) &= \sum_{i=1}^{m} \int_{\mathbb{R}} \chi_i(x)\, dx \\ &= \int_{\mathbb{R}} \sum_{i=1}^{m} \chi_i(x)\, dx \leqslant \int_G 2\, dx = 2\mathcal{H}^1(G) \leqslant 2(\mathcal{H}^1(F) + \epsilon), \end{aligned}$$

whence the claim (iii) follows. □ LEMMA 2.11

Analytic Implies Geometric

We next prove that analytically quasiconformal mappings quasi-preserve certain conformal invariants — in particular, capacities of rings.

2.12 **Theorem** ($\mathrm{QC_A} \Rightarrow \mathrm{QC_G}$). *If f is analytically K–quasiconformal in Ω, then*

$$\operatorname{Cap}(R) \leqslant K \operatorname{Cap}(f(R)) \tag{2.4}$$

for every ring R compactly contained in Ω.

PROOF Let $R \subset\subset \Omega$ be a ring with complementary components C_0 and C_1.[3] Suppose that u is a smooth admissible function for the image ring $f(R)$. The composition $u \circ f$ clearly satisfies the boundary conditions

$$(u \circ f)\big|_{C_0} \equiv 0 \quad \text{and} \quad (u \circ f)\big|_{C_1} \equiv 1.$$

Moreover, f is Sobolev by definition $\mathrm{QC_A}$, so it enjoys the ACL property in R, and consequently so does $u \circ f$. Therefore $u \circ f$ is admissible for R, and

[3]Recall that in our notation for rings, C_1 always stands for the bounded component and C_0 for the ubounded component.

by definition of capacity we have

$$\operatorname{Cap}(R) \leqslant \int_{\mathbb{R}^2} |\nabla(u \circ f)(x)|^2 \, dx. \tag{2.5}$$

By the hypothesis, f is differentiable at almost every point of Ω; since u is smooth, the chain rule is valid at these points:

$$\nabla(u \circ f)(x) = D^t f(x) \, (\nabla u \circ f)(x).$$

The derivative bound of Definition 2.1–$\mathrm{QC_A}$ thus implies

$$|\nabla(u \circ f)(x)|^2 \leqslant \|Df(x)\|^2 \, |(\nabla u \circ f)(x)|^2 \leqslant K \, Jf(x) \, |(\nabla u \circ f)(x)|^2 \tag{2.6}$$

at points of differentiability of f.

Homeomorphism f pulls back the Lebesgue measure via the rule $\nu(A) = |f(A)|$. The Radon–Nikodym Theorem states that the density $D\nu$ exists and is finite almost everywhere in R. Moreover, it satisfies

$$\int_B D\nu(x) \, dx \leqslant \nu(B) = \int_{f(B)} dy,$$

for every Borel set $B \subset \Omega$, and we infer that

$$\int_B (h \circ f)(x) \, D\nu(x) \, dx \leqslant \int_{f(B)} h(y) \, dy \tag{2.7}$$

holds for any nonnegative Borel-measurable function h. Here we denote $y = f(x)$.

At points of differentiability of f, the Radon–Nikodym derivative $D\nu$ coincides with the pointwise Jacobian Jf. Combining this with inequalities (2.5)–(2.7), we see that

$$\begin{aligned} \operatorname{Cap}(R) &\leqslant \int_R |\nabla(u \circ f)(x)|^2 \, dx \\ &\leqslant K \int_R Jf(x) \, |(\nabla u \circ f)(x)|^2 \, dx \\ &\leqslant K \int_{f(R)} |\nabla u(y)|^2 \, dy \end{aligned}$$

for an arbitrary $u \in \operatorname{Adm}_\infty(f(R))$. Taking the infimum of the right side over such u finishes the proof. □ Theorem 2.12

2.13 Remark. Observe that inequality (2.7) is a precursor to the change of variables formula for quasiconformal mappings. What is still missing to furnish the formula is Lusin's $\mathcal{N}$–property — the fact that quasiconformal maps preserve nullness of sets. We shall prove this later, in Chapter 5.

Geometric Implies Metric

We complete the cycle of implications by showing that if a mapping quasi-preserves capacities of rings, then its metric distortion is bounded. Part of the proof uses an idea from Pekka Koskela's notes [Kos09, page 7].

2.14 Theorem ($\mathrm{QC_G} \Rightarrow \mathrm{QC_M}$). *Let f be a homeomorphism on a planar domain Ω. If there exists a finite constant K such that*

$$\mathrm{Cap}(R) \leqslant K\ \mathrm{Cap}(f(R))$$

for every ring $R \subset\subset \Omega$, then f is metrically quasiconformal in Ω.

PROOF Let $x \in \Omega$. Given $r > 0$, let us abbreviate $L = L_r f(x)$ and $l = \ell_r f(x)$. Because f is homeomorphic, we can certainly find $d > 0$ such that $0 < d < \frac{1}{2}\operatorname{dist}(x, \partial\Omega)$ and for every $r \in (0, d)$ we have $\overline{\mathbb{D}}_L(f(x)) \subset f(\Omega)$. We fix such d and r, and denote $D_L = \overline{\mathbb{D}}_L(f(x))$ and $D_l = \overline{\mathbb{D}}_l(f(x))$. We put

$$C_L = f^{-1}(\mathbb{R}^2 \setminus D_L) \qquad \text{and} \qquad C_l = f^{-1}(D_l).$$

It is clear from this construction that both C_L and C_l meet the circle with centre x and radius r. Further, the ring $\mathcal{R}(\mathbb{R}^2 \setminus D_L, D_l)$ is contained in $f(\Omega)$ and its preimage is the ring $\mathcal{R}(C_L, C_l) \subset \Omega$, and so we have

$$\frac{1}{K}\,\mathrm{Cap}\left(\mathcal{R}(C_L, C_l)\right) \leqslant \mathrm{Cap}\left(\mathcal{R}(D_l, \mathbb{R}^2 \setminus D_L)\right) = 2\pi\left(\ln\frac{L}{l}\right)^{-1}, \qquad (2.8)$$

where the inequality holds by the hypothesis, and the equality is formula (1.10) for the capacity of an annulus.

Assertion of Theorem 2.14 will follow if we show that the ratio L/l has an upper bound in Ω. Thus by (2.8), it suffices to show that $\mathrm{Cap}(\mathcal{R}(C_L, C_l))$ is bounded away from zero. We establish this in the following lemma.

2.15 Lemma. *Suppose that a ring R separates the points x and y from the points z and ∞. If $|x - y| = |x - z| = r$, then there exists a constant C*

which is independent of r *and such that* $\operatorname{Cap}(R) \geqslant C$.

PROOF We may assume that x is the origin.

We can certainly find a ring $R' = \mathcal{R}(C_0, C_1)$ such that $R \subset R'$ and $y \in \partial C_1$, $z \in \partial C_0$. According to the assertion (i) of Theorem 1.20 on monotonicity of the capacity, $\operatorname{Cap}(R) \geqslant \operatorname{Cap}(R')$.

By the hypothesis, the points y and z lie on the circle $\mathbb{S}_r$. Depending on whether y and z happen to lie in the same quarter-circle or not, we choose

$$w = \begin{cases} \dfrac{y+z}{|y-z|}\, r, & \text{if an arc between } x \text{ and } y \text{ is shorter than } \dfrac{\pi r}{2}, \\[2ex] \dfrac{z}{2} & \text{otherwise.} \end{cases}$$

Next, we observe that whenever $\frac{\pi r}{4} < t < r$, the circle $\mathbb{S}_t(w)$ meets both complementary components C_0 and C_1. Therefore, every $u \in \operatorname{Adm}(R')$ varies between 0 and 1 on $\mathbb{S}_t(w)$, and Hölder's inequality tells us that

$$1 \leqslant \int_{\mathbb{S}_t(w)} |\nabla u| \leqslant \sqrt{2\pi t} \left(\int_{\mathbb{S}_t(w)} |\nabla u|^2 \right)^{\frac{1}{2}}.$$

We now obtain a lower bound for the capacity $\operatorname{Cap}(R')$ by virtue of

$$\int_{\mathbb{D}_{2r}} |\nabla u|^2 \geqslant \int_{\frac{\pi r}{4}}^{r} \left(\int_{\mathbb{S}^1_t(w)} |\nabla u|^2 \right) dt \geqslant \int_{\frac{\pi r}{4}}^{r} \frac{dt}{2\pi t} = \frac{1}{2\pi} \ln \frac{4}{\pi}.$$

This bound on the right — denote it by C — is independent of r, and the proof is complete. □ LEMMA 2.15

Returning to the proof of Theorem 2.14, we apply the above lemma to $R = \mathcal{R}(C_L, C_l)$. Then (2.8) implies that

$$\frac{L_r f(x)}{\ell_r f(x)} \leqslant e^{\frac{2\pi K}{C}} \tag{2.9}$$

for small enough radii r, and we conclude that the distortion Hf is uniformly bounded in Ω—that is to say, f is metrically quasiconformal. □ THEOREM 2.14

2.16 Remark. Essential to establishing a lower bound for $\operatorname{Cap}(\mathcal{R}(C_0, C_1))$ is the fact that the ring $\mathcal{R}(C_0, C_1)$ separates points x and y from both z

and point at infinity.[4] During the proof, we extracted certain local data on the distortion of a mapping from a global description of its geometric behaviour. In order to do this, we had to require that radius r — and the rings associated with it — is small enough to begin with (this is akin to the "egg yolk principle," see Theorem 3.6). In the next chapter, we will see how the method of Lemma 2.15 can be made systematic by employing symmetrization and the so-called extremal rings of Grötzsch and Teichmüller.

[4]As the proof shows, all such rings satisfying $|x - y| = |x - z|$ have the same lower bound for capacity, regardless of their size!

Chapter 3

Geometric Behaviour

In this chapter, we shall see that local geometric behaviour of quasiconformal mappings is characterized by the so-called quasisymmetries. Roughly speaking, quasisymmetries are to QC mappings what similarities are to conformal maps. Characterization via quasisymmetries is particularly useful for establishing the pseudogroup and compactness properties of quasiconformal maps.

Quasisymmetric Mappings

Recall that by a similarity transformation we mean a composition of scaling, rotation and translation. The linearization of a conformal map is a similarity transformation; for a quasiconformal mapping, a similar role is played by a quasisymmetry.[1]

3.1 Definition. Let $\eta\colon [0,\infty) \to [0,\infty)$ be a continuous strictly increasing function which satisfies $\eta(0) = 0$ and $\lim_{t\to\infty} \eta(t) = \infty$. Let f be a mapping on a domain Ω. We say that f is η*−quasisymmetric* in Ω if the inequality

$$\frac{|f(x) - f(y)|}{|f(x) - f(z)|} \leqslant \eta\left(\frac{|x-y|}{|x-z|}\right) \tag{3.1}$$

holds for any three distinct points x, y, and z in Ω.

This definition entails that a quasisymmetry f is continuous in Ω. To see this, fix points x and $z \neq x$ in Ω at random; the continuity at x follows by the fact that $\eta(t)$ goes to 0 with t.

[1]The term "quasisymmetry" originates from the geometric properties of such functions in one real variable. In higher dimensions, the term is somewhat misleading. These mappings could rather be named along the lines of "quasisimilarities" or "almost affine" transformations.

Comparing the reciprocals of both sides of (3.1) reveals that f is injective. Thus it admits an inverse f^{-1} defined on $f(\Omega)$.

Further, we can relabel the points in $\{x, y, z\}$, whence (3.1) is equivalent to

$$\frac{|x-y|}{|x-z|} \leqslant \frac{1}{\eta^{-1}\left(\dfrac{|f(x)-f(z)|}{|f(x)-f(y)|}\right)}.$$

Therefore f is η–quasisymmetric if and only if f^{-1} is θ–quasisymmetric with $\theta(t) = \frac{1}{\eta^{-1}(1/t)}$. By the above, f^{-1} is necessarily continuous; thus f is homeomorphic in Ω.

Observe that f is a similarity if inequality (3.1) is fulfilled by $\eta = \mathrm{id}$, the identity function, in which case (3.1) becomes

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} = \frac{|x-y|}{|x-z|}.$$

Lastly, it is a straightforward consequence of the definition that an η_1–quasisymmetric map followed by an η_2–quasisymmetric map is $(\eta_2 \circ \eta_1)$–quasisymmetric.

We gather these observations below.

3.2 Proposition. *Let f be an η–quasisymmetric mapping on Ω. The following claims hold:*

(*i*) *f is a homeomorphism;*

(*ii*) *the inverse mapping $f^{-1} : f(\Omega) \to \Omega$ is θ–quasisymmetric with*

$$\theta(t) = \frac{1}{\eta^{-1}(1/t)};$$

(*iii*) *f is a similarity transformation if η can be taken to be the identity map;*

(*iv*) *the class of quasisymmetric mappings is closed under composition.*

Claim (*iii*) of Proposition 3.2 links quasisymmetries and similarities. The following consideration sheds more light on the geometry of the former. Let f be η–quasisymmetric. Definition 3.1 tells us that

$$\frac{1}{\eta(1)} \leqslant \frac{|f(x)-f(y)|}{|f(x)-f(z)|} \leqslant \eta(1)$$

for all distinct points y and z belonging to a circle centred at x. This means that the image of the circle lies inside an annulus whose module (cf. 1.16) is controlled by the value of η at $t = 1$. Informally speaking, quasisymmetries send round sets to 'almost round' sets. This hints at a link with quasiconformal mappings: the latter send infinitesimal circles to ellipses of uniformly bounded ellipticity (definitions $\mathrm{QC_M}$ and $\mathrm{QC_A}$) and distort ring domains — but not too much (definition $\mathrm{QC_G}$). The following lemma is similar in spirit to the above remarks.

3.3 Proposition. *If f is η–quasisymmetric in Ω, then there is a constant $C = C(\eta)$, which depends only on η, such that*

$$\operatorname{diam}(f(D))^2 \leqslant C\,|f(D)|$$

for every disc D in Ω.

Rewriting the above inequality as

$$\frac{|f(D)|}{\operatorname{diam}(f(D))^2} \geqslant \frac{1}{C}$$

elucidates the idea that a disc's image under a quasisymmetry cannot be arbitrarily slender — $f(D)$ takes up a relatively large portion (area-wise) of the smallest disc containing it. This statement may be viewed as a global version of asserting the uniform boundedness of the metric distortion Hf.

PROOF Let $D \subset \Omega$ be a disc with centre x and radius r. Recall that we defined stretchings of f as

$$L_r f(x) = \max_{|h|=r} |f(x+h) - f(x)| \quad \text{and} \quad \ell_r f(x) = \min_{|h|=r} |f(x+h) - f(x)|.$$

From the definitions of the diameter of a set, quasisymmetry, and from elementary geometry we deduce the following bounds:

$$\operatorname{diam}(f(D)) \leqslant 2\,L_r f(x), \quad L_r f(x) \leqslant \eta(1)\,\ell_r f(x), \quad \pi\left(\ell_r f(x)\right)^2 \leqslant |f(D)|.$$

Combining these inequalities and setting

$$C(\eta) = \frac{4}{\pi}\eta(1)^2$$

yields the desired result. □ LEMMA 3.3

The following three results are the central topic of this chapter; they allow us to characterize QC mappings by quasisymmetries (at least locally).

3.4 Theorem. *If f is η–quasisymmetric in Ω, then f is $\eta(1)$–quasiconformal in Ω.*

Proof The most direct path lies via the metric Definition 2.1–QC_M.[2] Indeed, the distortion bound follows at once from Definition 3.1 of quasisymmetry, for we have

$$\frac{L_r f(x)}{\ell_r f(x)} = \frac{\max_{|h|=r} |f(x+h) - f(x)|}{\min_{|h|=r} |f(x+h) - f(x)|} \leqslant \eta\,(1)$$

whenever $0 < r < \operatorname{dist}(x, \partial\Omega)$, and so

$$Hf(x) = \limsup_{r \to 0} \frac{L_r f(x)}{\ell_r f(x)} \leqslant \eta(1) < \infty.$$

We note that this argument works in higher dimensions just as well. □ Theorem 3.4

The converse to Theorem 3.4 is not true in general, but it does hold for mappings that are globally quasiconformal.

3.5 Theorem. *If f is K–quasiconformal in $\mathbb{R}^2$, then it is η–quasisymmetric for some η depending only on K.*

Furthermore, the so-called 'egg-yolk principle' takes place: a mapping that is quasiconformal in Ω (possibly a strict subdomain of the plane) is quasisymmetric in a subset which lies 'deep enough' in Ω. In other words, quasisymmetries capture the local geometric behaviour of quasiconformal maps.

3.6 Theorem. *If f is K–quasiconformal in Ω, then every point of Ω lies in a neighbourhood U such that the restriction $f\big|_U$ is η–quasisymmetric for some η which depends on K.*

The remainder of this chapter is devoted to proving the two theorems

[2] Definition QC_M, while being intuitive and accessible, is generally not the most convenient to work with. In most settings, authors prefer to use the analytic or geometric definition of quasiconformality. Theorem 3.4 is a fortunate exception.

above. We chose to take a geometric approach, inspired by [GMP17, Section 6.6.2], because it links the ring capacity to the ancient isoperimetric problem, and — once the prerequisites are in place — yields the actual proof in an elegant manner while affording valuable geometric insight.

We first discuss an important inequality which asserts that the Dirichlet energy of a function decreases under symmetric rearrangements.

Symmetrization and a Rearrangement Inequality

A *rearrangement transformation* generally means a measure-preserving geometric manipulation of a set. A particularly useful class of such transformations comprises *symmetric rearrangements*, or *symmetrizations* for brevity; their utility manifests in the fact that they decrease certain set functions (but not set sizes). For instance, since Antiquity geometers have known that the most symmetric planar shape, the circle, realizes the smallest perimeter among all shapes enclosing a given area. In Geometric Measure Theory, this principle is known as the Isoperimetric Inequality. It turns out that ring capacity is subject to the same principle (Theorem 3.9), as a consequence of the Pólya–Szegő inequality (Theorem 3.8). The latter in fact generalizes the Isoperimetric Inequality.

We now give a definition of Pólya's circular symmetrization and list some of its basic properties. Proving these propreties would be too much of a digression; we refer the reader to Vladimir Dubinin's comprehensive book [DK14] on the topic.

3.7 Definitions. Let E be an open set in the plane and let L be a ray emanating from a point x at an angle φ. The *circular symmetrization* of E with respect to L is the set E^* defined by the following properties.

(*i*) If $E \cap \mathbb{S}_t(x)$ is empty or a full circle, then so is $E^* \cap \mathbb{S}_t(x)$, respectively.

(*ii*) If $E \cap \mathbb{S}_t(x)$ is a union of circular arcs amounting to angular measure ϑ, then $E^* \cap \mathbb{S}_t(x) = \{te^{i\theta} \colon \varphi - \vartheta/2 < \theta < \varphi + \vartheta/2\}$.

The symmetrization F^* of a closed set F is defined in a similar manner.

(*i*) If $F \cap \mathbb{S}_t(x)$ is empty or a full circle, then so is $F^* \cap \mathbb{S}_t(x)$, respectively.

(*ii*) If $F \cap \mathbb{S}_t(x)$ is a union of circular arcs amounting to angular measure ϑ, then $F^* \cap \mathbb{S}_t(x) = \{te^{i\theta} \colon \varphi - \vartheta/2 \leqslant \theta \leqslant \varphi + \vartheta/2\}$. Should $F \cap \mathbb{S}_t(x)$ be a countable union of singletons, then $F^* \cap \mathbb{S}_t(x) = L \cap \mathbb{S}_t(x)$.

The symmetrization of an open set is open, and the symmetrization of a closed set is closed. For a complete discussion of this and many other properties and applications of symmetrization, see [DK14, Chapter 4] or [GMP17, Section 5.3].

The definition of a symmetrized set entails that if $A \subset B$ then also $A^* \subset B^*$. However, the symmetrization of a doubly connected domain is not doubly connected unless the ray of symmetry originates in the 'hole.' We thus define the *symmetrization of a ring* R with the bounded complementary component C_1 by

$$R^* = (R \cup C_1)^* \setminus C_1^*,$$

which ensures that the symmetrization of a ring is also a ring.

Given a real-valued function u in $\mathbb{R}^2$, the *symmetrized function* u^* is constructed by simultaneous symmetrization of the super-level sets

$$D_t = \left\{x \in \mathbb{R}^2 \colon u(x) > t\right\}.$$

More precisely, at each point x we set

$$u^*(x) = \sup\left\{t \in \mathbb{R} \colon x \in D_t^*\right\}.$$

With the above definitions in place, we present the following celebrated inequality.

3.8 The Pólya–Szegő Inequality. *The Dirichlet energy of a function decreases under symmetric rearrangements:*

$$\int_{\mathbb{R}^2} |\nabla u^*|^2 \leqslant \int_{\mathbb{R}^2} |\nabla u|^2.$$

This result relies on concepts from the Geometric Measure Theory and several technical properties of symmetrization. A concise and accessible proof using the Isoperimetric Inequality in the spirit of the original work by Pólya and Szegő is presented in Giorgio Talenti's paper [Tal76, Lemma 1]. A more elementary but somewhat technical argument relying on complex variables can be found in Walter Hayman's book [Hay94, Chapter 4]

The following consequence is what is interesting in our context.

3.9 Theorem. *If R is a ring and R^* is the circular symmetrization of R, then*

$$\operatorname{Cap}(R^*) \leqslant \operatorname{Cap}(R).$$

In order to prove Theorem 3.9, we appply the Pólya–Szegő inequality to admissible functions of R. However, we need to ensure that the symmetrization u^* of an admissible function $u \in \mathrm{Adm}(R)$ is itself admissible of class $\mathrm{Adm}(R^*)$. We shall only present a technical lemma which says that the symmetrization of a function is 'at least as continuous' as the function itself. This lemma corresponds to Example 4.1 in [Hay94].

3.10 Lemma. *Let u be a function and let u^* be its symmetrization. Define*

$$E = \{z \colon a \leqslant u(z) \leqslant b\} \qquad \text{and} \qquad E^* = \{z \colon a \leqslant u^*(z) \leqslant b\},$$

and define

$$L_\delta = \sup_{\substack{x,y\in E \\ |x-y|\leqslant\delta}} |u(x) - u(y)| \qquad \text{and} \qquad L_\delta^* = \sup_{\substack{x,y\in E^* \\ |x-y|\leqslant\delta}} |u^*(x) - u^*(y)|.$$

Suppose that E is bounded and $L_\delta < \infty$. Under this hypothesis, $L_\delta^* \leqslant L_\delta$.

The quantity L_δ is called *modulus of continuity* of u on E, and the condition $L_\delta \to 0$ as $\delta \to 0$ says that u is uniformly continuous on E. The proof below only uses elementary geometry of the plane.

Proof We may assume that u is symmetrized with respect to the positive real half-line. Suppose, towards a contradiciton, that $L_\delta^* > L_\delta$. Then we can find points $z_1 = (r_1, \theta_1)$ and $z_2 = (r_2, \theta_2)$ in E^*, such that

$$|z_1 - z_2|^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2) < \delta^2 \tag{3.2}$$

and

$$u^*(z_1) - u^*(z_2) > L_\delta. \tag{3.3}$$

Note that (3.2) is equivalent to the system

$$|r_1 - r_2| \leqslant \delta \qquad \text{and} \qquad \cos(\theta_1 - \theta_2) \geqslant \frac{r_1^2 + r_2^2 - \delta^2}{2r_1r_2}. \tag{3.4}$$

By (3.3), we can select numbers t_1 and t_2 such that

$$a \leqslant u^*(z_2) < t_2 < t_1 - L_\delta < u^*(z_1) - L_\delta \leqslant b - L_\delta. \tag{3.5}$$

Let us define

$$\vartheta(r,t) = \{\theta \in [-\pi, \pi) \colon u(r,\theta) > t\},$$

and let $|\vartheta(r,t)|$ denote the angular Lebesgue measure of the set $\vartheta(r,t)$. Then, (3.5) shows that $z_1 \in D^*_{t_1}$, but $z_2 \notin D^*_{t_2}$, whence

$$2|\theta_1| < |\vartheta(r_1,t_1)| \qquad \text{and} \qquad 2|\theta_2| > |\vartheta(r_2,t_2)|.$$

Thus

$$|\theta_2 - \theta_1| \geqslant \big||\theta_2| - |\theta_1|\big| > \frac{1}{2}\big||\vartheta(r_2,t_2)| - |\vartheta(r_1,t_1)|\big|. \tag{3.6}$$

We obtain a lower bound for the quantity on the right via the following considerations. Observe that $\theta_2 \in \vartheta(r_2,t_2)$ if $\theta_1 \in \vartheta(r_1,t_1)$ and $r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2) < \delta^2$ (otherwise there are two points closer than δ on which u takes values t_1 and t_2, which differ by more than L_δ). We see, therefore, that $\vartheta(r_1,t_1) \subset \vartheta(r_2,t_2)$, and neither of these sets is the entire range of angles $[-\pi,\pi)$, because at least $z_2 \notin D^*_{t_2}$. Further, for every $\theta' \in \vartheta(r_1,t_1)$ we can say that $\theta \in \vartheta(r_2,t_2)$ if

$$\theta' - \arccos\frac{r_1^2 + r_2^2 - \delta^2}{2r_1r_2} \leqslant \theta \leqslant \theta' + \arccos\frac{r_1^2 + r_2^2 - \delta^2}{2r_1r_2},$$

and hence

$$|\vartheta(r_2,t_2)| - |\vartheta(r_1,t_1)| > 2\arccos\frac{r_1^2 + r_2^2 - \delta^2}{2r_1r_2}. \tag{3.7}$$

Combining (3.6) and (3.7), we obtain

$$\cos(\theta_2 - \theta_1) < \frac{r_1^2 + r_2^2 - \delta^2}{2r_1r_2},$$

which is in contradiction with (3.4). □ LEMMA 3.10

Extremal Rings of Grötzsch and Teichmüller

Theorem 3.9 lets us find a lower bound for the capacity of a ring by comparing the latter with the so-called extremal rings; these rings can be thought of as the 'most symmetric' rings out there.

3.11 Definition. For $r > 1$ and $s > 1$ we define the *Grötzsch ring* $\mathcal{R}_G(r)$ and the *Teichmüller ring* $\mathcal{R}_T(s)$ by putting

$$\mathcal{R}_G(r) = \mathcal{R}(\overline{\mathbb{D}},\, [r,\infty]),$$
$$\mathcal{R}_T(s) = \mathcal{R}([0,1],\, [s,\infty]).$$

With the extremal rings we associate functions Φ and Ψ determined by

$$\begin{aligned} \ln\Phi(r) &= \operatorname{Mod}(\mathcal{R}_G(r)), & \Phi\colon (1,\infty] &\to (1,\infty]; \\ \ln\Psi(s) &= \operatorname{Mod}(\mathcal{R}_T(s)), & \Psi\colon (1,\infty] &\to (1,\infty]. \end{aligned}$$

As the module is preserved under similarity transformations, we will hereafter use the same terminology when referring to the scaled, rotated, and translated versions of $\mathcal{R}_G(r)$ and $\mathcal{R}_T(s)$. For example, if $r' > r > 1$ and $s' > s > 1$, then the (scaled) Grötzsch ring $G = \mathcal{R}(\overline{\mathbb{D}}_r,\, [r',\infty])$ and the (scaled) Teichmüller ring $T = \mathcal{R}([0,s],\, [s',\infty])$ satisfy

$$\operatorname{Mod}(G) = \ln\Phi\left(\frac{r'}{r}\right) \qquad \text{and} \qquad \operatorname{Mod}(T) = \ln\Phi\left(\frac{s'}{s}\right).$$

The rings defined above are unbounded. There is also a bounded variant of the Grötzsch ring that finds its use. Specifically, conformal inversion in the unit circle[3] transforms $\mathcal{R}_G(r)$ into the bounded ring $\mathcal{R}([0,1/r],\, \mathbb{R}^2 \setminus \mathbb{D})$ which has the same module $\ln\Phi(r)$.

Note that the functions Φ and Ψ are both strictly increasing on $(1,\infty]$ and satisfy

$$\begin{aligned} \lim_{r\to 1}\Phi(r) &= 1, & \lim_{r\to\infty}\Phi(r) &= \infty, \\ \lim_{s\to 1}\Psi(s) &= 1, & \lim_{s\to\infty}\Psi(s) &= \infty, \end{aligned}$$

by property (i) of Theorem 1.20. By property (ii) of the same theorem, they are continuous.

The following theorem is a consequence of Theorem 3.9; it justifies the word 'extremal' in the name of the ring domains defined above. Inequalities (3.8) and (3.9) below are known as the Grötzsch and Teichmüller estimate, respectively. Informally speaking, these inequalities are a step forward from the assertion of Lemma 2.15.

3.12 Theorem. *If a ring R_1 separates the disc $\mathbb{D}_r(x)$ from the points z and ∞, then*

$$\operatorname{Mod}(R_1) \leqslant \ln\Phi\left(\frac{|x-z|}{r}\right), \tag{3.8}$$

[3] Known as the Kelvin transform $z \mapsto 1/\bar{z}$.

and if a ring R_2 separates points x and y from z and ∞, then

$$\mathrm{Mod}(R_2) \leqslant \ln \Psi\left(\frac{|x-z|}{|x-y|}+1\right). \tag{3.9}$$

PROOF Let R_1^* be the ring obtained by symmetrizing R_1 with respect to the ray emanating from x in the direction opposite from z. This ring R_1^* contains a Grötzsch ring which is similar to the Grötzsch ring of radius $|x-z|/r$. The inequality (3.8) follows by combining monotonicity of the capacity (assertion (i) of Theorem 1.20) with Theorem 3.9. An identical argument is used to prove (3.9). □ THEOREM 3.12

Applying the above theorem to a Grötzsch ring inverted in a circle (cf. Defintion 3.11) yields the following result.

3.13 **Corollary.** *If a ring R separates the circle $\mathbb{S}_r(x)$ from the points x and $z \in \mathbb{D}_r(x)$, then*

$$\mathrm{Mod}(R) \leqslant \ln \Phi\left(\frac{r}{|x-z|}\right). \tag{3.10}$$

Quasiconformality Implies Quasisymmetry

3.14 **Proving Theorems 3.5 and 3.6** Now we have all the geometric tools necessary to construct a homeomorphism η.

We follow two main sources in the proofs below. The CASE 1 in the first proof is inspired by the geometric approach in [GMP17, Section 6.6.2]. The two remaining cases are taken from [AIM08, p. 69]. We additionally provide an alternative solution to CASE 3 in Appendix B; this other proof uses the same ring module technique as in CASE 1, but relies on the fact that the inverse of a quasiconformal map is quasiconformal, which is yet to be proved in the next chapter.

Let us momentarily outline the central idea for the ring module approach. The problem at hand is to bound the distortion exerted by the family of all K–quasiconformal transformations on a fixed triangle. The idea is to express the aspect ratio of a triangle in terms of the module of some ring separating two of triangle's vertices from the third vertex — and vice versa (this is why functions Φ and Ψ appear in the triple inequality in CASE 1 below).

Recall that the geometric Definition 2.1–$\mathrm{QC_G}$ is equivalent to this: f is

K–quasiconformal in Ω if there exists a finite nonzero constant K such that

$$\mathrm{Mod}(f(R)) \leqslant \frac{1}{K}\,\mathrm{Mod}(R) \tag{3.11}$$

for every ring $R \subset\subset \Omega$.

PROOF OF THEOREM 3.5 There are three cases to consider.

CASE 1 Suppose that $|x-y| > |x-z|$. We may as well assume that $|f(x)-f(y)| > |f(x)-f(z)|$, because otherwise (3.1) holds trivially: any increasing homeomorphism η with $\eta(1) \geqslant 1$ will do. Consider the Grötzsch ring that separates the disc of radius $|f(x)-f(z)|$ centred at $f(x)$ and the ray emanating from $f(y)$ in the direction opposite from $f(x)$. Let us call this ring R', and let its preimage under f be R. Now, R separates points x and z from y and ∞, so the Teichmüller estimate (3.9) applies. Together with property (3.11), this entails

$$\ln\Phi\left(\frac{|f(x)-f(y)|}{|f(x)-f(z)|}\right) = \mathrm{Mod}(R') \leqslant \frac{1}{K}\,\mathrm{Mod}(R) \leqslant \frac{1}{K}\,\ln\Psi\left(\frac{|x-y|}{|x-z|}+1\right),$$

whence

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \leqslant \Phi^{-1}\left(\Psi\left(\frac{|x-y|}{|x-z|}+1\right)^{\frac{1}{K}}\right). \tag{3.12}$$

We see that f is satisfies inequality (3.1) for any η which dominates the function

$$\Theta(t) = \Phi^{-1}\left(\Psi(t+1)^{\frac{1}{K}}\right), \qquad t > 1, \tag{3.13}$$

which is monotone increasing, tends to infinity with t, and has a finite limit greater or equal to 1 as $t \to 1$ from above.

CASE 2 The other case, when $|x-y| < |x-z|$, turns out to be a bit more subtle.

Assume $|x-y| = r \leqslant R = |x-z|$ and fix an arbitrary $\varepsilon > 0$. Using the fact that f is a homeomorphism, we infer that

$$|f(x)-f(y)| \leqslant L_r f(x) < L_{R+\varepsilon} f(x) = \sup_{|x-z'|=R+\varepsilon} |f(x)-f(z')|.$$

From the outcome of Case 1, we know that

$$\frac{|f(x)-f(z')|}{|f(x)-f(z)|} \leqslant \Theta\left(\frac{R+\varepsilon}{R}\right)$$

for every z' such that $|x-z'| = R+\varepsilon$. Combining the two inequalities above, together with the fact that ε was chosen at random, we conclude that

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \leqslant \Theta(1). \tag{3.14}$$

As the bound on the right is independent of r and R, we have just proved that f is *weakly H–quasisymmetric*; this means that there exists a finite nonzero constant H (equal to $\Theta(1)$ in our case), such that

$$|x-y| \leqslant |x-z| \qquad \text{implies} \qquad |f(x)-f(y)| \leqslant H\,|f(x)-f(z)| \tag{3.15}$$

for all triples x, y, z of distinct points in Ω.

Weak quasisymmetry of course gives an upper bound on the quotient $|f(x)-f(y)|/|f(x)-f(z)|$, but no continuous real function η can simultaneously satisfy $\lim_{t\to 0}\eta(t) = 0$ and dominate $H > 1$ on $(0,1)$. Therefore, we have to carefully study what happens when $|x-y|/|x-z|$ approaches zero.

Case 3 Without loss of generality, we may assume $a^n\,|x-y| = |x-z|$ for some $a \in [3,9]$ and $n \in \mathbb{N}$.

Let us first make the following observation. Fix a number $R > 0$ and suppose that the points x, y, w satisfy $|x-y| \leqslant R$ and $|x-w| = 2R$. Let w' be the midpoint of the line segment $[x,w]$, and let $w'' \neq w'$ be some arbitrary point such that $|w-w''| = R$. Applying the weak quasisymmetry condition (3.15) thrice gives

$$\begin{aligned}|f(x)-f(y)| &\leqslant H\,|f(x)-f(w')| \\ &\leqslant H^2\,|f(w')-f(w)| \leqslant H^3\,|f(w)-f(w'')|,\end{aligned}$$

and since w'' was selected at random, we conclude that

$$\pi\,|f(x)-f(y)|^2 \leqslant H^6\,\pi\,\big(\ell_R f(w)\big)^2 \leqslant H^6\,|f(\mathbb{D}_R(w))|, \tag{3.16}$$

or, in other words, $f(\mathbb{D}_R(w))$ contains a disc of radius $H^{-3}\,|f(x)-f(y)|$.

Now, by the hypothesis on the points x, y, z, we can certainly find at least n disjoint discs $D_j = \mathbb{D}_{R_j}(w_j) \subset \mathbb{D}_{a^n}(x)$ that satisfy

$$2\,|x-y| \leqslant 2R_j = |x-w_j|, \qquad j = 1, \ldots, n.$$

Because the discs D_j have disjoint images under the homeomorphism f, and in view of (3.16), we infer that

$$\begin{aligned} n\pi\,|f(x)-f(y)|^2 \leqslant H^6 \sum_{j=1}^{n} |f(D_j)| \leqslant H^6\,|f(\mathbb{D}_{a^n}(w))| \\ \leqslant H^6\,\pi\left(L_{a^n} f(x)\right) \leqslant \pi H^8\,|f(x)-f(z)|^2. \end{aligned}$$

But by construction,

$$\frac{1}{2}\log_3 \frac{|x-z|}{|x-y|} \leqslant n,$$

and so f satisfies inequality (3.1) for every η dominating the function

$$\widetilde{\Theta}(t) = \Theta(1)^4 \left(\frac{2\ln 3}{\ln(1/t)}\right)^{\frac{1}{2}}, \qquad 0 < t < \frac{1}{3}, \tag{3.17}$$

which is monotone increasing and goes to zero with t.

Now we can certainly construct a homeomorphism of $[0, \infty)$ onto itself which dominates all three bounds (3.13), (3.14), and (3.17) on respective intervals, which renders f quasisymmetric. □ THEOREM 3.5

PROOF OF THEOREM 3.6 For $a \in \Omega$, we shall use notations

$$r'_a = \operatorname{dist}\Big(f(a), \partial f(\Omega)\Big) \qquad \text{and} \qquad r_a = \operatorname{dist}\Big(a, \partial f^{-1}\big(\mathbb{D}_{r'_a}(f(a))\big)\Big).$$

Fix $x \in \Omega$. Let y and z be distinct points in the disc $\mathbb{D}_{r_x}(x)$. Next, we argue as in the proof of Theorem 3.5, with slight modification. In the case when $|x-y| > |x-z|$ and $|f(x)-f(y)| > |f(x)-f(z)|$, let R' be the bounded Grötzsch ring separating the line segment $[f(x), f(z)]$ from the circle $\mathbb{S}_{|f(x)-f(y)|}(f(x))$. The preimage $f^{-1}(R')$ separates x and z from y and ∞; the Teichmüller estimate (3.9) and formula (3.11) imply that the bound (3.12) holds. If, on the other hand, $|x-y| < |x-z|$, we argue exactly as in the previous proof to obtain the same bounds as in (3.14) and (3.17).

This shows that every point $x \in \Omega$ lies in a neighbourhood $\mathbb{D}_{r_x}(x)$ such that the inequaltiy

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \leqslant \eta\left(\frac{|x-y|}{|x-z|}\right) \tag{3.18}$$

holds for any pair of distinct points $y, z \in \mathbb{D}_{r_x}(x)$. This is not enough to conclude that f is quasisymmetric because, for obvious reasons, x is not some arbitrary point in $\mathbb{D}_{r_x}(x)$. However, we argue that x has a sufficiently small neighbourhood in which f is quasisymmetric. In particular, define

$$U = \bigcap_u \left\{\mathbb{D}_{r_u}(u)\colon u \in \mathbb{D}_{r_x/3}(x)\right\},$$

so if $u, v, w \in U$, then $u, v, w \in \mathbb{D}_{r_u}(u) \cap \mathbb{D}_{r_v}(v) \cap \mathbb{D}_{r_w}(w)$. Using the triangle inequality one can show that U is a nonempty neighbourhood of x. The claim follows, as the bound (3.18) holds for any three arbitrary points $x, y, x \in U$ by construction of the latter. $\square$ THEOREM 3.6

Chapter 4

Distortion in the Complex Plane

So far we have discussed four various characterizations of quasiconformal mappings: the three Definitions 2.1 and the quasisymmetry characterization. All of these descriptions extend to an arbitrary dimension greater than two. The language of complex variables lends yet another remarkable desription of planar quasiconformal mappings in terms of solutions of a particular partial differential equation, which is the subject of the final chapter. In this chapter, we set the ground for the hard work ahead.

A Background in Complex Analysis

Recall that we identify a complex number z with an element (x, y) of the real plane by virtue of writing $z = x + iy$. For a complex function f, we conventionally write $f = u + iv$, where u and v are real-valued mappings of the plane. Despite $\mathbb{C}$ and $\mathbb{R}^2$ being isomorphic as real vector spaces, they possess very different notions of differentiability.

4.1 Complex differentiability. Let $U \subset \mathbb{C}$ be open. We say that a complex function $f\colon U \to \mathbb{C}$ is *complex-differentiable* at a point $z \in U$ provided the limit

$$f'(z) = \lim_{\zeta \to 0} \frac{f(z+\zeta) - f(z)}{\zeta} \tag{4.1}$$

exists, in which case we call $f'(z)$ the *complex derivative* of f at z. Of course (4.1) is equivalent to

$$\lim_{\zeta \to 0} \frac{f(z+\zeta) - f(z) - f'(z)\,\zeta}{\zeta} = 0, \tag{4.2}$$

from which it follows immediately that a mapping is complex-differentiable if and only if it is real-differentiable with a complex-linear differential. In particular, the complex derivative $f'(z)$ — viewed as a differential — is the $\mathbb{C}$–linear transformation $\zeta \mapsto f'(z)\,\zeta$, amounting to rotation and scaling. The real differential of $f = u + iv$ at $z = x + iy$ is given by the matrix

$$\begin{bmatrix} u_x(z) & u_y(z) \\ v_x(z) & v_y(z) \end{bmatrix},$$

and so f is complex-differentiable at z if and only if it is real-differentiable at z and the equations

$$u_x(z) = v_y(z), \qquad u_y(z) = -v_x(z), \tag{4.3}$$

called the *Cauchy–Riemann equations*, hold.

Furthermore, if the complex differential $f'(z)$ exists and is not zero, then f is conformal at z in the sense of Definition 1.8.

We say that f is *holomorphic* in U if $f'(z)$ exists at every point $z \in U$. We may also say that f is holomorphic at a point if there is a disc centred at that point in which f is holomorphic. If, in addition, the complex derivative f' is zero-free in U, then f is conformal in U.

4.2 Analytic functions. We say that $f\colon U \to \mathbb{C}$ is *analytic* at $z \in U$ if it is representable by a convergent power series in some open disc around z; that is, if there exists a *radius of convergence* $r > 0$ such that $|\zeta - z| < r$ implies

$$f(\zeta) = \sum_{k=0}^{\infty} a_k(\zeta - z)^k.$$

A function is analytic in U if it is analytic at every point z in U.

If a function is analytic at $z \in U$, with the radius of convergence $r > 0$, then its power series expansion is complex-differentiable infinitely often in $\mathbb{D}_r(z)$; consequently f is holomorphic in $\mathbb{D}_r(z)$. One of the basic theorems in complex function theory is the converse claim: if a function is holomorphic, then it is analytic. The classical Cauchy integral formula provides exact formulæ for computing the coefficients of the power series expansion.

If f is holomorphic in a *punctured domain* $\Omega \setminus \{z_0\}$, then we say that the point z_0 is an *isolated singularity* of f. A useful tool when studying functions with singularities are the formal *Laurent series*

$$\sum_{k=-\infty}^{\infty} a_k(\zeta - z)^k = \sum_{n=0}^{\infty} a_n(\zeta - z)^n + \sum_{n=0}^{\infty} \frac{a_{-n}}{(\zeta - z)^n}.$$

The first series on the right is called the *analytic part* of the Laurent series, while the remaining terms constitute the *principal part.*

An isolated singularity can be one of three kinds: a *removable singularity*, an *essential singularity*, or a *pole.* The point z is a removable singularity if the principal part of the Laurent series expansion of f vanishes, in which case f is actually analytic in all of Ω. The point z is an essential singularity if the coefficients $a_{-n} \neq 0$ for infinitely many indices n. Finally, the point z is a pole of order m if $a_{-n} = 0$ for all $n > m$. If z is a pole of order $m = 1$, we call z a *simple pole.*

4.3 Behaviour of functions at ∞. The *value of f at infinity*, denoted by the symbol $f(\infty)$, is to be interpreted as $\lim_{z\to\infty} f(z)$, if such limit exists (that is, if its value is independent of the direction of approaching ∞). In the same spirit we write $f(z_0) = \infty$ with the obvious interpretation that $\lim_{z\to z_0} f(z) = \infty$. The latter implies that f has a non-removable singularity at z_0.

In order to study the behaviour of a function at infinity, we conjugate it with the inversion in the complex plane. In particular, we say that

- $f(z)$ is continuous (resp. differentiable) at ∞ if $z \mapsto f(1/z)$ is continuous (resp. differentiable) at 0;

- for a function such that $f(z_0) = \infty$, f is continuous (resp. differentiable) at z_0 if $z \mapsto 1/f(z)$ is continuous (resp. differentiable) at z_0.

In our study, we will often encounter homeomorphisms of $\mathbb{C}$ onto itself. If $f\colon \mathbb{C} \to \mathbb{C}$ is a homeomorphism, we can normalize it so that $f(\infty) = \infty$. In view of the above, we say that such f is continuous (resp. differentiable) at ∞ if the conjugate map

$$z \mapsto \frac{1}{f(1/z)}$$

is continuous (resp. differentiable) at 0.

We say that f is *analytic in the neighbourhood of infinity* if there is an annulus $\mathcal{A}(R, \infty)$ — a punctured neighbourhood of ∞ — on which f can be developed into the Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k, \qquad R < |z| < \infty.$$

We have $f(\infty) = \infty$ only if the analytic part does not vanish. On the other hand, f is differentiable at ∞ only if the analytic part has finitely many nonzero terms, which is equivalent to f having a pole at ∞. Lastly, f is analytic and injective near infinity if and only if it is of the form

$$f(z) = a_1 z + a_0 + \sum_{k=1}^{\infty} \frac{a_{-k}}{z^k}, \qquad a_1 \neq 0.$$

We define $f'(\infty) = a_1$ for such a function. Because every Möbius transformation that fixes ∞ restricts to a similarity on $\mathbb{C}$, f admits the Laurent series expansion

$$f(z) = z + \sum_{k=1}^{\infty} \frac{a_{-k}}{z^k} \tag{4.4}$$

for large z, after suitable normalization. Representation formula (4.4) is crucial in deriving the formulæ for the principal solutions of the Beltrami equation in §§6.11–6.13.

4.4 Complex notation. We introduce two differential operators

$$\partial = \frac{1}{2}\left(\partial_x - i\,\partial_y\right) \qquad \text{and} \qquad \bar\partial = \frac{1}{2}\left(\partial_x + i\,\partial_y\right), \tag{4.5}$$

known in the literature as the *Wirtinger derivatives*. If f is a complex function whose real and imaginary parts have partial derivatives, we shall also use the notation $f_z = \partial f$ and $f_{\bar z} = \bar\partial f$.

Wirtinger derivatives have numerous applications and are ubiquitous in the geometric function theory. For instance, the Cauchy–Riemann system of equations (4.3) becomes one simple expression:

$$\bar\partial f = 0. \tag{4.6}$$

Equation (4.6) is known as the homogeneous $\bar\partial$–equation.[1]

Having introduced the operators in (4.5), we define their counterpart

[1] The symbol $\bar\partial$ is pronounced "dee-bar."

differential forms

$$dz = dx + i\,dy \qquad \text{and} \qquad d\bar{z} = dx - i\,dy.$$

When integrating with respect to the two-dimensional Lebesgue measure, we will use the symbol $|dz|^2$ to denote an area element $dx \wedge dy$ located at $z = x + iy$. Note that $dz \wedge d\bar{z} = (dx - i\,dy) \wedge (dx + i\,dy) = (-2i)\,dx \wedge dy$, and so we can interpret the wedge product $dz \wedge d\bar{z}$ as an area element of imaginary measure. We thus adopt the notations

$$\int f(z)\,|dz|^2 = \int f(z)\,dx \wedge dy = -\frac{1}{2i} \int f(z)\,dz \wedge d\bar{z}$$

for the 2-dimensional area integral of the complex function f. The motivation is to be able to distinguish the area integral from the line integral, which we will denote by

$$\int f(z)\,dz.$$

4.5 Green's Theorem in complex form.. *Let Ω be a bounded domain with piecewise continuously differentiable boundary. If P and Q are in $W^{1,1}(\Omega) \cap C(\overline{\Omega})$, then*

$$\int_{\partial\Omega} P(\zeta)\,d\zeta + Q(\zeta)\,d\bar{\zeta} = \int_{\Omega} \big(Q_z(\zeta) - P_{\bar{z}}(\zeta)\big)\; d\zeta \wedge d\bar{\zeta} \tag{4.7}$$

$$= -2i \int_{\Omega} \big(Q_z(\zeta) - P_{\bar{z}}(\zeta)\big)\; |d\zeta|^2. \tag{4.8}$$

Here, $P_{\bar{z}}$ and Q_z are weak derivatives, and the identities hold in the sense of distributions.

We only outline the main idea of the proof.

Proof sketch For a general continuously differentiable 1-form $\omega = \alpha\,dz + \beta\,d\bar{z}$, we compute

$$\begin{aligned} d\omega &= \partial\omega + \bar{\partial}\omega \\ &= (\alpha_z\,dz \wedge dz + \beta_z\,dz \wedge d\bar{z}) + (\alpha_{\bar{z}}\,d\bar{z} \wedge dz + \beta_{\bar{z}}\,d\bar{z} \wedge d\bar{z}) \\ &= (\beta_z - \alpha_{\bar{z}})\,dz \wedge d\bar{z}. \end{aligned} \tag{4.9}$$

The generalized Stokes' Theorem asserts that the form ω satisfies

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega. \tag{4.10}$$

Combining (4.9) with (4.10) furnishes the Green's formula for the case when $P, Q \in C^1(\overline{\Omega})$. The general case follows by a mollification argument. □ THEOREM 4.5

Next we present one of the fundamental results of all geometric function theory — the generalized Cauchy formula — using the complex derivatives introduced in § 4.4. As is the case with the classical Cauchy–Goursat formula, this result is a consequence of the Green's Theorem of multivariate calculus, which is in turn the Stokes' Theorem in the plane.

4.6 **Generalized Cauchy Formula.** *Let Ω be a bounded domain with piecewise continuously differentiable boundary. If f belongs to $W^{1,1}(\Omega) \cap C(\overline{\Omega})$, then*

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{\pi}\int_{\Omega} \frac{f_{\bar z}(\zeta)}{\zeta - z}\, |d\zeta|^2 \tag{4.11}$$

for every z in Ω.

We observe that if f is analytic in Ω, then $f_{\bar z}$ vanishes, and formula (4.11) reduces to the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

familiar from the course in classic complex analysis.

PROOF We show the validity of (4.11) for $f \in C^1$. The general case follows by approximation via mollification.

Fix a $z \in \Omega$ and for $\zeta \neq z$ define

$$P(\zeta) = \frac{f(\zeta)}{\zeta - z}.$$

Select a radius $\epsilon > 0$ small enough so that the disc $\mathbb{D}_\epsilon(z)$ lies in Ω and apply Green's formula (4.7) to P on the domain $\Omega \setminus \overline{\mathbb{D}}_\epsilon(z)$ to obtain

$$\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int_{\partial\mathbb{D}_\epsilon(z)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = -\int_{\Omega\setminus\overline{\mathbb{D}}_\epsilon(z)} \frac{f_{\bar z}(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar\zeta. \tag{4.12}$$

We compute the second integral on the left to be

$$\int_{\partial\mathbb{D}_\epsilon(z)} \frac{f(\zeta)\,d\zeta}{\zeta - z} = \int_0^{2\pi} \frac{f(z+\epsilon e^{it})}{\epsilon e^{it}}\, i\epsilon e^{it}\,dt = i\int_0^{2\pi} f(z+\epsilon e^{it})\,dt,$$

and the last expression tends to $2\pi i\, f(z)$ as we let $\epsilon \to 0$. Substituting this into (4.12) completes the proof. □ THEOREM 4.6

Complex Dilatation

In this section, we formulate the distortion of a planar mapping by means of identifying $\mathbb{R}^2$ with $\mathbb{C}$.

4.7 **Definition.** Let f be a complex function defined in Ω. We define the *complex dilatation* of f by

$$\mu_f(\zeta) = \frac{f_{\bar z}(\zeta)}{f_z(\zeta)}, \tag{4.13}$$

at points $\zeta \in \Omega$ where the Wirtinger derivatives of f exist and $f_z(\zeta) \neq 0$.

Let us now look at some examples.

4.8 **Linear mappings.** Let T be a planar linear transformation with the real-matrix representation

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Observe that T can be written as

$$T(z) = Az + B\bar z, \tag{4.14}$$

where the complex coefficients A and B are given by

$$A = \frac{1}{2}\big((a+d) - i(b-c)\big), \qquad B = \frac{1}{2}\big((a-d) + i(b+c)\big). \tag{4.15}$$

A direct calculation also shows that the determinant of T is given by

$$\det(T) = |A|^2 - |B|^2. \tag{4.16}$$

Suppose now that T is nonsingular and sense-preserving. We express the distortion of T in terms of A and B. The image of every point z on the unit circle $\mathbb{S}^1$ satisfies an estimate

$$||A| - |B|| \leqslant |T(z)| \leqslant |A| + |B|,$$

and each bound is attained in the respective principal direction. Since T is sense-preserving and nonsingular, we see from (4.16) that $|A| - |B| > 0$. In view of the discussion in § 1.2, the distortion of T is given by

$$H(T) = \frac{|A| + |B|}{|A| - |B|}. \tag{4.17}$$

For a more geometric view, we introduce the complex dilatation

$$\mu = \frac{B}{A}$$

of T, which lets us write

$$T(z) = A(z + \mu\bar{z}). \tag{4.18}$$

Multiplication by complex number A amounts to rotation and scaling only, so all distortion must come from the map $z \mapsto z + \mu\bar{z}$, and thus can be expressed in terms of the complex dilatation μ alone. Indeed, we can see at once that the distortion in (4.17) can be written in the form

$$H(T) = \frac{1 + |\mu|}{1 - |\mu|}. \tag{4.19}$$

4.9 Differentiable and conformal mappings. Let $f = u + iv$ be real-differentiable at $\zeta \in \mathbb{C}$. The differential of f can be written as a matrix of partial derivatives,

$$Df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix},$$

and we combine (4.14)–(4.16) with (4.5) to obtain

$$Df(\zeta)\, z = f_z(\zeta)\, z + f_{\bar{z}}(\zeta)\, \bar{z}, \tag{4.20}$$

and also

$$Jf(\zeta) = |f_z(\zeta)|^2 - |f_{\bar{z}}(\zeta)|^2. \tag{4.21}$$

Suppose now that f is a sense-preserving homeomorphism and suppose additionally that $Df(\zeta)$ is nonsingular. It follows that $|f_z(\zeta)| - |f_{\bar{z}}(\zeta)| > 0$. Theorem 1.6, combined with (4.17), (4.19), implies that the infinitesimal distortion of f at at ζ is given by

$$Hf(\zeta) = \frac{|f_z(\zeta)| + |f_{\bar{z}}(\zeta)|}{|f_z(\zeta)| - |f_{\bar{z}}(\zeta)|} = \frac{1 + |\mu_f(\zeta)|}{1 - |\mu_f(\zeta)|}. \tag{4.22}$$

If, in addition, f happens to be complex-differentiable (and thus conformal) at ζ, then the Cauchy–Riemann equations imply $f_{\bar{z}}(\zeta) = 0$, and so the differential $Df(\zeta) : z \mapsto f_z(\zeta)\, z$ is a complex-linear transformation, as we expect. We see at once that a conformal map has vanishing complex dilatation and unit distortion. On the other hand, the converse is true on the strength of Weyl's Lemma 4.12 below.

4.10 Chain rule for the Wirtinger derivatives. The operators ∂ and $\bar{\partial}$ obey the chain rule. Observe that (4.5) can be written formally in the matrix form:

$$\begin{bmatrix} \partial h \\ \bar{\partial} h \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} (D^t h)$$

for a mapping h. Noting that

$$I = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix},$$

and using the chain rule for $D(g \circ f)$, we write

$$\begin{aligned} \begin{bmatrix} \partial (g \circ f) \\ \bar{\partial} (g \circ f) \end{bmatrix} &= \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} (D^t f)\, I\, (Dg \circ f)^t \\ &= \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} (D^t f) \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} \partial g \circ f \\ \bar{\partial} g \circ f \end{bmatrix}. \end{aligned}$$

Performing somewhat tedious matrix multiplication on the right side, we find that

$$\begin{bmatrix} \partial (g \circ f) \\ \bar{\partial} (g \circ f) \end{bmatrix} = \begin{bmatrix} \partial f & \overline{\bar{\partial} f} \\ \bar{\partial} f & \overline{\partial f} \end{bmatrix} \begin{bmatrix} \partial g \circ f \\ \bar{\partial} g \circ f \end{bmatrix}, \tag{4.23}$$

which is the chain rule for the Wirtinger derivatives.

4.11 Lemma (Transformation rule for the complex dilatation). *Suppose $f \colon \Omega \to \Omega'$ and $g \colon \Omega' \to \Omega''$ are quasiconformal mappings. Define the com-*

plex dilatations

$$\mu_f = \frac{\bar\partial f}{\partial f}, \qquad \mu_{g\circ f} = \frac{\bar\partial(g\circ f)}{\partial(g\circ f)}, \qquad \mu_g \circ f = \frac{\bar\partial g\circ f}{\partial g \circ f}.$$

The identity

$$\mu_g \circ f = \frac{\mu_{g\circ f} - \mu_f}{1 - \mu_{g\circ f}\,\overline{\mu_f}}\,\frac{\partial f}{\overline{\partial f}} \tag{4.24}$$

holds at almost every point in Ω*.*

PROOF Formally inverting the matrix on the right side in (4.23) gives

$$\begin{aligned}\partial g\circ f &= \frac{1}{Jf}\big(\overline{\partial f}\cdot\partial(g\circ f) - \overline{\bar\partial f}\cdot\bar\partial(g\circ f)\big),\\ \bar\partial g\circ f &= \frac{1}{Jf}\left(\partial f\cdot\bar\partial(g\circ f) - \bar\partial f\cdot\partial(g\circ f)\right).\end{aligned}$$

As f is quasiconformal, $Jf(z) > 0$ at almost every $z\in\Omega$ by Corollary 5.4, and so the two identities above hold almost everywhere. Formula (4.24) follows from dividing the second expression by the first. □ LEMMA 4.11

A Lemma of Weyl

The following celebrated result shows that the assertion '$f_{\bar z} = 0$ if and only if f is analytic' holds in great generality. In the proof we follow unpublished notes by Kari Astala. For another elementary proof, see [Hub16, Theorem 4.1.6].

4.12 Weyl's Lemma. *Suppose* $f \in W^{1,1}_{\text{loc}}(\Omega)$*. The following are equivalent:*

(*i*) *f has a representative which is analytic in* Ω*;*

(*ii*) *$f_{\bar z} = 0$ in the distributional sense;*

(*iii*) *$f_{\bar z}(\zeta) = 0$ for almost every* $\zeta\in\Omega$*.*

PROOF It is clear that (*i*) implies (*ii*) and (*iii*). Also the equivalence of (*ii*) and (*iii*) is a basic fact about planar Sobolev functions. We prove that (*i*) follows from (*ii*) by considering convolution approximation of f. Specifically, letting φ_ϵ be the sequence of smooth bump mollifiers, each $\varphi_\epsilon * f$ is analytic in Ω_ϵ, since for $z \in \Omega_\epsilon$ we have

$$\bar{\partial}(\varphi_\epsilon * f)(z) = \bar{\partial} \int_\Omega \varphi_\epsilon(z-\zeta)\, f(\zeta)\, d\zeta = \int_\Omega \bar{\partial}\varphi_\epsilon(z-\zeta)\, f(\zeta)\, d\zeta = 0$$

by (ii).[2] Then $\varphi_\epsilon * f \to f$ locally in $W^{1,1}$ norm. If f happens to be continuous in Ω, the convergence is also uniform on every compact subset of Ω, whence the limit mapping is analytic. For a more general f, we observe that by the mean value property of analytic functions, we have for $z \in \Omega_{2\epsilon}$

$$(\varphi_\epsilon * f)(z) = \frac{1}{|D_z|} \int_{D_z} (\varphi_\epsilon * f)(\zeta)\, d\zeta,$$

where D_z is the disc with centre at z and radius $\frac{1}{2}\operatorname{dist}(z, \partial\Omega)$. Thus we have the bound

$$\big|(\varphi_\epsilon * f)(z) - (\varphi_{\epsilon'} * f)(z)\big| \leqslant \frac{1}{|D_z|} \int_{D_z} \big|(\varphi_\epsilon * f)(\zeta) - (\varphi_{\epsilon'} * f)(\zeta)\big|\, d\zeta,$$

and the right side goes to 0 with ϵ and ϵ' by the $L^1(D_z)$ convergence. We now see that $\varphi_\epsilon * f \to f$ uniformly on compact subsets of Ω, and $g = \lim_{\epsilon\to 0}(\varphi_\epsilon * f)$ is an analytic representative of f we were looking for. □ LEMMA 4.12

We now finally obtain this long anticipated result.

4.13 Corollary (to Weyl's Lemma). *A 1–quasiconformal mapping is analytic; being a homeomorphism, it is in fact conformal.*

PROOF From (4.22) and Definition 4.7 we see that if f is 1–quasiconformal, then $f_{\bar{z}} = 0$. Hence f is analytic by Weyl's Lemma 4.12. □ COROLLARY 4.13

[2]Here, it is crucial that $f_{\bar{z}} = 0$ *in the sense of distributions.* As a counterexample, the inversion $z \mapsto 1/z$ is locally integrable and satisfies condition (iii) of the lemma, but is not analytic in the unit disc.

Chapter 5

Essential Properties

In this chapter, we record some of the most useful — and sometimes unexpected — properties of quasiconformal mappings.

Pseudogroup Property

5.1 Theorem. *Quasiconformal mappings form a pseudogroup. More precisely,*

(*i*) *the inverse of a K–quasiconformal mapping is K–quasiconformal, and*

(*ii*) *the composition of a K–quasiconformal and K'–quasiconformal mapping is KK'–quasiconformal.*

Proof We already know from Chapter 3 that quasisymmetries form a pseudogroup. Applying successively Theorem 3.6, Proposition 3.2, and Theorem 3.4 to a K–quasiconformal mapping f, we see that f^{-1} is quasiconformal. It is not clear from the listed theorems whether the distortion bounds coincide. However, we observed in Remark 2.3 that this indeed must be the case, in view of the properties of ring distortions (a proof based on a direct computation can be found in [AIM08, Theorem 3.7.7]). This proves (*i*). Assertion (*ii*) follows by a similar reasoning. □ Theorem 5.1

Analytic Properties

By analytic properties we mean results which allow us to do calculus on quasiconformal mappings.

5.2 Theorem (Area formula). *If f is K–quasiconformal on Ω, then*

$$|f(B)| = \int_B Jf(x)\,dx \tag{5.1}$$

for every Borel measurable set $B \subset \Omega$.

We follow the argument of [AIM08, Theorem 3.3.7], which relies on the complex form of the Green's Theorem. Needless to say, this proof only works in the planar setting. Another proof via identification with quasisymmetric mappings is presented in [AIM08, Section 3.7].

PROOF To begin with, let $R \subset\subset \Omega$ be a rectangle with sides parallel to the axes. Because f is absolutely continuous on almost all lines parallel to the axes, we can enlarge R, if necessary, so that f is AC on ∂R (the integral over R does not change). Since $f\big|_{\partial R}$ is AC, the image $f(\partial R)$ of the boundary is rectifiable, and so we can apply the complex Green's Theorem 4.5 to $f(R)$. In particular, setting $P(\zeta) = \bar{\zeta}$ in formula (4.8), we have

$$|f(R)| = \int_{f(R)} 1\,|d\zeta|^2 = \int_{f(R)} P_{\bar{z}}(\zeta)\,|d\zeta|^2 = \frac{1}{2i}\int_{\partial f(R)} \bar{\zeta}\,d\zeta.$$

The integral on the right is a line integral of an absolutely continuous homeomorphism, so we are justified to change variables, whereby

$$|f(R)| = \frac{1}{2i}\int_{\partial R} \bar{f}(\tau)\,df(\tau).$$

Writing $df = f_z\,dz + f_{\bar{z}}\,d\bar{z}$ and applying Green's theorem (4.8) to R once again, we obtain

$$|f(R)| = \int_R \Big(|f_z(\tau)|^2 - |f_{\bar{z}}(\tau)|^2\Big)\,|d\tau|^2 = \int_R Jf(\tau)\,|d\tau|^2, \tag{5.2}$$

where the last equality is given by formula (4.21). Observe that we are permitted to apply Green's formula here because f is of class $W^{1,2}_{\mathrm{loc}}(\Omega)$ by the hypothesis, and hence the Jacobian Jf is locally integrable in Ω.

More generally, for a Borel subset $B \subset \Omega$, we may as well assume $\overline{B} \subset \Omega$ without loss of generality. For every $\epsilon > 0$ we can find an open set U_ϵ, such that $B \subset U_\epsilon \subset\subset \Omega$ and $|U_\epsilon \setminus B| < \epsilon$. Being an open set, U_ϵ is a union of countably many half-open disjoint rectangles with sides parallel to the axis, each satisfying (5.2). Hence, (5.2) is also valid for U_ϵ. The integral is locally absolutely continuous as a set function, and so

$$\int_B Jf(\tau)\,|d\tau|^2 = \lim_{\epsilon\to 0}\int_{U_\epsilon} Jf(\tau)\,|d\tau|^2 = \lim_{\epsilon\to 0}|f(U_\epsilon)| \geqslant |f(B)|.$$

By the Radon–Nikodym Theorem the reverse equality must hold, and so equality (5.1) follows. □ THEOREM 5.2

5.3 Corollary ($\mathcal{N}$–property). *If f is K–quasiconformal on Ω, then*

$$|B| = 0 \qquad \text{if and only if} \qquad |f(B)| = 0$$

for every Borel measurable set $B \subset \Omega$.

PROOF By the Radon–Nikodym Theorem, the quasiconformal homeomorphism f satisfies

$$\int_B Jf(x)\,dx \leqslant |f(B)|,$$

with equality occuring if and only if f satisfies the Lusin's $\mathcal{N}$–property: $|B| = 0$ implies $|f(B)| = 0$. The latter clearly holds on the validity of the area formula (5.1). As the inverse of a quasiconformal map is quasiconformal by Theorem 5.1, the reverse implication holds as well.

We remark that the restriction to Borel sets is necessary, because a homeomorphic image of a null set need not be null. □ COROLLARY 5.3

From the above results, we obtain the following immediate corollary.

5.4 Corollary (Positivity of the Jacobian). *The Jacobian determinant of a quasiconformal map is positive almost everywhere.*

5.5 Theorem (Change of variables). *If f is quasiconformal on Ω and $h \in L^1(f(\Omega))$, then*

$$\int_{f(\Omega)} h(y)\,dy = \int_\Omega (h\circ f)(x)\,Jf(x)\,dx. \tag{5.3}$$

PROOF By Theorem 5.2, formula (5.3) holds if h is a characteristic function of a Borel subset of Ω. If h is a nonnegative and integrable in $f(\Omega)$, the preimage of a null set under $h\circ f$ is null by Corollary 5.3 and therefore $h\circ f$ is measurable. We can thus find a sequence of simple functions $h_k \leqslant h$ on

$f(\Omega)$, converging almost everywhere to h and such that

$$\lim_{k\to\infty}\int_{\Omega}(h_k\circ f)(x)\,Jf(x)\,dx=\int_{f(\Omega)}h(y)\,dy.$$

Corollary 5.3 now entails that $u_k\circ f\to u$ almost everywhere in Ω, and the claim follows. □ THEOREM 5.5

5.6 **Theorem (Chain rule).** *If $f\colon\Omega\to\Omega'$ and $g\colon\Omega'\to\Omega''$ are quasiconformal mappings, then*

$$D(g\circ f)(x)=(Dg\circ f)(x)\,Df(x)$$

holds at almost every $x\in\Omega$.

PROOF By definition f and g are differentiable almost everywhere in Ω and Ω', respectively. As f preserves sets of measure zero, the composition $(g\circ f)$ is differentiable almost everywhere, and the classical chain rule applies. □ THEOREM 5.6

Convergence and Compactness

The class of quasiconformal mappings possesses much more flexibility than the conformal mappings. Quite surprisingly, quasiconformal mappings enjoy quite strong convergence properties. This is crucial for proving the Measurable Riemann Mapping Theorem which is the subject of the next chapter.

5.7 **Equicontinuity and normal families.** Let us recall some basic terminology.

We say that a family $\mathcal{F}$ of mappings defined in Ω is *equicontinuous* at $x_0\in\Omega$, if for every $\epsilon>0$ there exists $\delta>0$ such that $|x-x_0|<\delta$ implies $|f(x)-f(x_0)|<\epsilon$ for all $f\in\mathcal{F}$. If this holds true for every $x_0\in\Omega$, family $\mathcal{F}$ is said to be *equicontinuous.*

A family of continuous functions is called *normal* if every sequence contains a subsequence that converges locally uniformly to a limit function.

5.8 **Proposition.** *Suppose that a sequence of K–quasiconformal mappings on Ω converges uniformly to a homeomorphism on Ω in every compact subset of Ω. Under this hypothesis, the limit mapping is K–quasiconformal in Ω.*

This remarkable compactness result can be proved using the geometric definition of quasiconformality. For this, one would need a slight variation on part (*ii*) of Lemma 1.20; see Remark 1.21. Theorem 5.8 then follows rather directly from the geometric definition $\mathrm{QC}_{\mathrm{G}'}$. For an accessible description of this proof in three dimensions, see Theorem 3 in Frederick Gehring's paper [Geh62].

In our exposition, we follow [AIM08, Section 3.9] and present a proof based on the identification of QC maps with quasisymmetries. It is very easy to show that the latter form equicontinuous families. The theorem of Arzelà and Ascoli then guarantees that such families are normal. We make use of the following verison of this theorem.

5.9 The Arzelà–Ascoli Theorem. *Let $\Omega \subset \mathbb{C}$ be a domain and let $X = (X, d)$ be a complete metric space. A familiy of continuous maps $\Omega \to X$ is normal if and only if it is equicontinuous at every point in Ω.*

When studying families of functions that are locally bounded (that is, functions under which the images of compacta are uniformly bounded), it suffices that the target space in Theorem 5.9 is the finite complex plane equipped with the Euclidean metric, $X = (\mathbb{C}, |\cdot|)$.

5.10 Lemma. *Let Ω be a planar set containing distinct points a and b. The family of all η–quasisymmetric mappings on Ω that fix a and b is equicontinuous.*

Proof As similarity transformations are identity-quasisymmetric, we may assume that $a = 0$ and $b = 1$.

Let f be a mapping of the family described in the statement of the lemma. Fix a point x in Ω. For every $y \in \Omega$ we have

$$\frac{|f(x) - f(y)|}{|f(x) - f(1)|} \leqslant \eta\left(\frac{|x-y|}{|x-1|}\right), \qquad \frac{|f(x) - f(y)|}{|f(x) - f(0)|} \leqslant \eta\left(\frac{|x-y|}{|x|}\right).$$

In the cases when $x = 0$ or $x = 1$, the bound $|f(x) - f(y)| \leqslant \eta(|x-y|)$ holds independently of f.

On the other hand, we have

$$|f(x)| \leqslant \frac{|f(x) - f(0)|}{|f(1) - f(0)|} \leqslant \eta(|x|),$$

and, consequently, the bounds

$$|f(x) - f(1)| = |f(x) - 1| \leqslant |f(x)| + 1 \leqslant \eta(|x|),$$
$$|f(x) - f(0)| = |f(x) - 0| \leqslant |f(x)| \leqslant \eta(|x|),$$

which hold independent of f. This proves the lemma. □ LEMMA 5.10

We may relax the assumption that every mapping in the family of the previous lemma satisfies $f(a) = a$ and $f(b) = b$ by pre- and post-composing with similarities. The only additional requirement is a uniform bound on the moduli $|f(a)|$ and $|f(b)|$. We thus arrive at the following corollary.

5.11 **Corollary.** *Every locally bounded family of η–quasisymmetric mappings on Ω is normal. In particular, the limit of a sequence of η–quasisymmetries is either η–quasisymmetric or constant.*

We conclude this section with a compactness result which will prove invaluable at the end of the next chapter.

5.12 **Theorem.** *Every family of K–quasiconformal homeomorphisms of $\mathbb{C}$ that fix the points 0 and 1 is a normal family. Further, every limit mapping is a non-constant K–quasiconformal homeomorphism of the Riemann sphere $\overline{\mathbb{C}}$ that fixes the points 0 and 1.*

We remark that every quasisymmetry f a priori has normalization at infinity, in the sense that $\lim_{x\to\infty} f(x) = \infty$. For general families of quasiconformal homeomorphisms of $\overline{\mathbb{C}}$, we may need to impose an additional normalization so that three points in $\overline{\mathbb{C}}$ are fixed. This of course can still be achieved by pre- and post-composing with Möbius transformations.

PROOF By the results of the previous chapter, every limit mapping f of a convergent sequence $\{f_k\}_{k\in\mathbb{N}}$ of normalized K–quasiconformal mappings is quasiconformal. We only need to show that it satisfies the distortion bound

$$\|Df(x)\|^2 \leqslant K\, Jf(x) \tag{5.4}$$

almost everywhere.

Because the convergence is uniform on compact sets, the derivatives Df_k converge to Df weakly in $L^2(D)$ for every disc $D \subset \mathbb{C}$. By the weak lower semicontinuity of the L^2 norm and continuity of the Lebesgue measure, we have

$$\begin{aligned}\int_D \|Df(x)\|^2 dx &\leqslant \liminf_{k\to\infty} \int_D \|Df_k(x)\|^2 dx \\ &\leqslant K \liminf_{k\to\infty} \int_D Jf_k(x)\, dx = K \lim_{k\to\infty} |f_k(D)| = K\,|f(D)|\end{aligned}$$

for every disc D. Lebesgue's Density Theorem now implies that the inequalty (5.4) indeed holds at almost every point. $\square$ THEOREM 5.12

Chapter 6

Existence Theory

The goal of this chapter is to demonstrate that the K–quasiconformal planar mappings are identified with the solutions of the partial differential equation

$$f_z = \mu\, f_{\bar{z}},$$

where $\mu\colon \mathbb{C} \to \mathbb{C}$ is measurable and satisfies $\|\mu\|_{L^\infty} = \frac{K-1}{K+1} < 1$.

Beltrami's Equation

6.1 Definitions. Let $\mu\colon \Omega \to \mathbb{C}$ be measurable and satisfies $\|\mu\|_{L^\infty(\Omega)} < 1$. We interpret the definition (4.13) of the complex dilatation as the partial differential equation

$$f_{\bar{z}} = \mu\, f_z, \tag{BE}$$

called the *Beltrami equation* associated with the *Beltrami coefficient* μ.

We say that a function f is a *generalized solution* of the Beltrami equation in Ω, if f is ACL in Ω and the complex derivatives f_z and $f_{\bar{z}}$ satisfy (BE) at almost every point in Ω. We shall see later that it is natural to look for the solutions in the Sobolev class $W^{1,2}_{\text{loc}}(\Omega)$ of functions whose derivatives satisfy (BE) in the distributional sense.

Observe that should the coefficient μ vanish, Beltrami's equation (BE) reduces to the $\bar{\partial}$– (Cauchy–Riemann) equation (4.6). If f has continuous classical derivatives f_z and $f_{\bar{z}}$ that satisfy (BE) everywhere, then f is conformal outside the support of μ. Similarly, if f is Sobolev on compact sets and (BE) holds in the distributional sense, then f has an analytic representative outside the support of μ. This is asserted by Lemma 4.12 of Weyl.

6.2 Homeomorphic solutions and quasiconformality. We shall now show that the class of *homeomorphic* generalized solutions of the Beltrami equation (BE) is identified with the class of quasiconformal mappings whose complex dilatation equals μ in a set of full measure.

First, let g be a K–quasiconformal mapping on Ω. By the analytic definition, g is of class $W^{1,2}_{\mathrm{loc}}(\Omega)$ (and hence ACL) and differentiable almost everywhere in Ω. Moreover, the differential of g is nonsingular almost everywhere, and thus the quotient $\frac{g_{\bar z}}{g_z}$ exists at almost every point of Ω. Moreover, this quotient defines a measurable function on the domain Ω, as it is the ratio of two derivatives of a continuous function. The distortion condition $Hg(z) = H(Dg(z)) \leqslant K$, which holds at almost every $z \in \Omega$, by (4.22) can be written in the form

$$\left|\frac{g_{\bar z}(z)}{g_z(z)}\right| \leqslant \frac{K-1}{K+1}.$$

We thus see that g is a generalized solution of the Beltrami equation (BE) with $\mu = \frac{g_{\bar z}}{g_z}$.

Conversely, suppose that g is a homeomorphic generalized solution of (BE) in Ω. As a planar ACL homeomorphism g is differentiable almost everywhere, and the complex dilatation μ_g of g exists almost everywhere in Ω and trivially $\mu_g(z) = \mu(z)$ at almost every $z \in \Omega$. We can certainly find $K \geqslant 1$ such that

$$\frac{K-1}{K+1} = \|\mu\|_{L^\infty(\Omega)} < 1, \tag{6.1}$$

and so, in view of (4.22), g satisfies the distortion bound $Hg \leqslant K$ at points where μ exists. Thus two of the three requirements of the analytic definition $\mathrm{QC_A}$ are satisfied and g is either K–quasiconformal or a K–quasiconformal map composed with a reflection. In either case, the Jacobian Jg is non-zero almost everywhere in Ω. However, by (4.21) and the fact that $\|\mu\|_{L^\infty(\Omega)} < 1$ we have

$$Jg(z) = |g_z(z)|^2 - |g_{\bar z}(z)|^2 = |g_z(z)|^2\big(1 - |\mu(z)|^2\big) \geqslant 0,$$

and we conclude that $Jg(z) > 0$ at almost every $z \in \Omega$ and g is sense-preserving. It follows that g is K–quasiconformal in Ω.

Let us gather our observations.

(*i*) *Every K–quasiconformal mapping defined on Ω is a generalized solution of the Beltrami equation* (BE) *where μ coincides with the complex dilatation of said quasiconformal mapping (up to a null set).*

(*ii*) *Conversely, if $\mu\colon \Omega \to \mathbb{C}$ is measurable and satisfies* (6.1)*, then every* *homeomorphic generalized solution of the Beltrami equation* (BE) *in* Ω *is a* K*–quasiconformal mapping on* Ω*.*

Essentially, statement (i) is the analytic definition of quasiconformal mappings in disguise. The second claim is not too far removed from the assertion of the Measurable Riemann Mapping Theorem, which we now present in its full generality.

6.3 The Measurable Riemann Mapping Theorem. *Let* $\mu\colon \Omega \to \mathbb{C}$ *be a measurable function such that* $\|\mu\|_{L^\infty(\Omega)} = k < 1$*. There exists a* $\frac{1+k}{1-k}$*–quasiconformal mapping* $f\colon \mathbb{C} \to \mathbb{C}$ *which is a generalized solution of the Beltrami equation*

$$f_{\bar z} = \mu\, f_z, \tag{BE}$$

in Ω*. In particular,* f *is a sense-preserving homeomorphism of class* $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ *and* $f_{\bar z}(z) = \mu(z)\, f_z(z)$ *holds at almost every* $z \in \Omega$*.*

The assertion holds true also when $\Omega = \mathbb{C}$*, in which case the solution* f *is a quasiconformal homeomorphism of* $\overline{\mathbb{C}}$*, satisfies* $f(\infty) = \infty$*, and is unique up to post-composing with a similarity transformation. Moreover, the solution that fixes the points* 0 *and* 1 *is unique.*

PROOF STRUCTURE The proof we present in the sequel is long and can appear somewhat mazy for the uninitiated. Here is a roadmap for what follows. The assumption $\|\mu\|_{L^\infty} = k < 1$ stands throughout.

1. We introduce two singular integral operators $\mathcal{C}$ and $\mathcal{S}$ (§§6.6–6.10).

   First off, $\mathcal{C}$ has the property $\mathcal{C} = \bar\partial^{-1}$ on $C^1_\circ$ and $\mathcal{S}$ is an isometry of L^2 satisfying $\mathcal{S} \circ \bar\partial = \partial$.

   We also verify that $\bar\partial \circ \mathcal{C} = \mathrm{id}$ and $\partial \circ \mathcal{C} = \mathcal{S}$ hold on the space L^2 in the sense of distributions.

2. Assuming μ has compact support, we use the operators $\mathcal{C}$ and $\mathcal{S}$ to find that (BE) is solvable in the homogeneous Sobolev class $\dot W^{1,2} = \{f \in W^{1,2}_{\mathrm{loc}} \colon f_z, f_{\bar z} \in L^2\}$ (Theorem 6.11).

   By Weyl's Lemma 4.12, every generalized solution f of (BE) must be analytic off the support of μ. Normalizing if necessary, we may assume that f decays rapidly enough at infinity, that is, f has a representative

   $$f(z) = z + g(z), \qquad \text{with} \quad g \in \mathcal{O}\left({}^1\!/_z\right),$$

   whereby the weak derivatives $f_z = 1 + g_z$ and $f_{\bar z} = g_{\bar z}$ are globally in L^2.The operator $\mathcal{S}\colon L^2 \to L^2$ satisfies $\mathcal{S} \circ \bar\partial = \partial$. This allows us to translate (BE) into an integral equation:

   $$\bar\partial f = \mu\, \partial f \quad \rightsquigarrow \quad (\mathrm{id} - \mu\mathcal{S})\, g_{\bar z} = \mu.$$

The latter is solvable for $g_{\bar{z}}$ in L^2 because $\mathcal{S}$ is an isometry. What is more, $g_{\bar{z}} \in L^2_\circ$ because the support of μ is compact. The integral operator $\mathcal{C}\colon L^2_\circ \to \dot{W}^{1,2}$ allows us to recover the solution f from $g_{\bar{z}}$ $(= f_{\bar{z}})$.

3. The L^2 theory for the operators $\mathcal{C}$ and $\mathcal{S}$ is not enough to infer homeomorphicity of solutions. We still work with compactly supported μ.

   For continuity of $\mathcal{C}g_{\bar{z}}$, we need higher integrability of $g_{\bar{z}}$ (§§ 6.14–6.15).

   The kernel of $\mathcal{C}$ is too singular for the convolution with a general $\phi \in L^p$, $p \geqslant 2$, to converge. We alter the definition of $\mathcal{C}$ slightly, thus giving $\mathcal{C}\phi$ the meaning of an equivalence class of functions modulo an additive constant. The property $\mathcal{C} = \bar{\partial}^{-1}$ still holds though.

   For all $2 < p < \infty$ we have $\mathcal{C}\colon L^p \to C^\alpha$ with $\alpha = 1 - 2/p$ (C^α stands for the class of α–Hölder continuous functions). Further, $\mathcal{S}\colon L^p \to L^p$ acts continuously and the L^p–operator norm $\|\mathcal{S}\|_p$ is continuous in p. Hence we can find $p > 2$ such that $\|\mu\mathcal{S}\| \leqslant k\|\mathcal{S}\|_p < 1$, and $(\mathrm{id} - \mu\mathcal{S})$ is, therefore, invertible. The integral equation from the previous step is thus solvable for $g_{\bar{z}}$ in L^p and the principal solution $f = z + \mathcal{C}g_{\bar{z}}$ is Hölder continuous.

   We now fix $p > 2$ such that $k\|\mathcal{S}\|_p < 1$ and show that principal solutions are continuously differentiable in $\mathbb{C}$ whenever the weak derivative μ_z belongs to L^p (Theorem 6.20).

   If $\mu_z \in L^p$, then the function $\sigma_{\bar{z}} = (\mathrm{id} - \mu\mathcal{S})^{-1}\mu_z$ is in L^p and $\sigma = \mathcal{C}\sigma_{\bar{z}}$ is in C^α, by the above. The function $F(z) = z + \mathcal{C}(\mu e^\sigma)(z)$ is then in C^1 and its weak derivatives satisfy $F_z = e^\sigma$ and $F_{\bar{z}} = \mu\, e^\sigma$ simultaneously. Therefore, F is a generalized C^1 solution of (BE).

   Further, a principal solution is necessarily a homeomorphism of the Riemann sphere $\overline{\mathbb{C}}$ (§ 6.22).

   From the formula $F(z) = z + \mathcal{C}(\mu e^\sigma)(z)$ we find that the Jacobian $JF > 0$ strictly everywhere, and so F is locally homeomorphic in $\mathbb{C}$. A suitable normalization ensures that $\sigma \in \mathcal{O}(1/z)$ and $F \in \mathcal{O}(z)$ at ∞. Thus we can conformally extend F to $\overline{\mathbb{C}}$ by putting $F(\infty) = \infty$. By the Monodromy theorem, F is then a global homeomorphism of $\mathbb{C}$.

   In light of the discussion in § 6.2, principal solutions are thus quasiconformal (Theorem 6.23).

4. In the final steps, we use compactness properties of quasiconformal mappings (Theorem 5.12) to relax the assumptions on μ.

□

Uniqueness of the Solutions

Even before finding the solutions, we can see that it is a simple matter to establish the uniqueness of the solutions of the Beltrami equation — with the help of the transformation formula for the complex dilatation and Weyl's Lemma.

6.4 Lemma. *Let f and g be quasiconformal in Ω. The following are equivalent.*

(*i*) $\mu_f(z) = \mu_g(z)$ *at almost every* $z \in \Omega$.

(*ii*) $g = \phi \circ f$ *for some* ϕ *conformal in* Ω.

PROOF To see why (*ii*) implies (*i*), observe that because ϕ is smooth, the chain rule (4.23) applies. By conformality $\bar\partial \phi \equiv 0$, and hence $\partial g = \partial\phi\,\partial f$, $\bar\partial g = \partial\phi\,\bar\partial f$, and $\mu_g = \mu_f$ almost everywhere.

For the reverse implication, we use the fact that f is a homeomorphism, and we can define $\phi = g \circ f^{-1}$. As a composition of quasiconformal maps, ϕ is quasiconformal by Theorem 5.1 and we use Lemma 4.11 to write

$$\mu_\phi \circ f = \frac{\mu_g - \mu_f}{1 - \mu_g\,\overline{\mu_f}}\,\frac{\partial f}{\overline{\partial f}}.$$

By the hypothesis the quantity on the right vanishes at almost every point in Ω, and so by Corollary 4.13 to Weyl's lemma, ϕ is conformal. □ LEMMA 6.4

In the special case $\Omega = \mathbb{C}$, the corollary above immediately yields the following uniqueness result, since the only conformal mappings of the complex plane onto itself are similarities.

6.5 Corollary (Uniqueness part of the MRMT). *If f and g are $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ homeomorphic solutions of the same Beltrami equation in $\mathbb{C}$, then*

$$g = \phi \circ f$$

for some similarity transformation $\phi\colon z \mapsto az + b$, with $a, b \in \mathbb{C}$.

Integral Operators $\mathcal{C}$ and $\mathcal{S}$

Formulæ (6.5) and (6.9) below express the properties that make the Cauchy and Beurling transforms a useful tool for solving the Beltrami equation.

6.6 The Cauchy transform $\mathcal{C}$. We denote the class of continuously differentiable complex functions with compact support in $\mathbb{C}$ by the symbol $C^1_\circ(\mathbb{C})$.

On this space, the *Cauchy transform* is defined by

$$(\mathcal{C}\phi)(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\phi(\zeta)}{\zeta - z}\,|d\zeta|^2, \qquad \phi \in C^1_\circ(\mathbb{C}). \tag{6.2}$$

Here and in the sequel, singular integrals are to be understood in the *Cauchy principal value* sense, that is

$$(\mathcal{C}\phi)(z) = -\frac{1}{\pi}\lim_{\epsilon\to 0}\int_{|\zeta - z|>\epsilon} \frac{\phi(\zeta)}{\zeta - z}\,|d\zeta|^2, \tag{6.3}$$

The Cauchy transform is a convolution operator with the kernel

$$\mathcal{K}(\tau) = -\frac{1}{\pi\tau},$$

called the *Cauchy kernel.*

The characteristic feature of the Cauchy transform is implied by the generalized Cauchy formula (4.11), which shows that

$$(\mathcal{C}\phi_{\bar z})(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\phi_{\bar z}(\zeta)}{\zeta - z}\,|d\zeta|^2 = \phi(z), \tag{6.4}$$

or, in other words, the Cauchy transform is the left inverse of the $\bar\partial$ operator on the class $C^1_\circ(\mathbb{C})$.[1] Being a convolution operator, the Cauchy transform clearly commutes with translations; it therefore also commutes with all first order linear partial differential operators on the class $C^1_\circ$, including $\bar\partial$. We write this consisely as

$$\mathcal{C}\circ\bar\partial = \bar\partial\circ\mathcal{C} = \mathrm{id} \qquad \text{on the space } C^1_\circ(\mathbb{C}). \tag{6.5}$$

6.7 The Beurling transform $\mathcal{S}$. The operator $\mathcal{S}$ is most easily understood as a Fourier multiplier. Let us first recall some basics of the Fourier transform.

The Fourier transform $\mathcal{F}$ on the space $L^1(\mathbb{C})$ is defined by the formula

$$(\mathcal{F}f)(\zeta) = \int_{\mathbb{C}} f(z)\, e^{-2\pi i(x\xi + y\eta)}\,dx\,dy,$$

where we write $z = x + iy$, $\zeta = \xi + i\eta$. If $f \in L^1 \cap L^2$, then $\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2}$, and the Fourier transform extends to an isometry of $L^2(\mathbb{C})$ by Plancherel's theorem.[2] If f and the distributional derivatives f_x and f_y all belong to

[1] Rudin in [Rud87], p. 389, provides an elementary proof of (6.4) that does not rely on Green or Stokes type results.

[2] More than that: this extension of $\mathcal{F}$ is a unitary operator of L^2 .

$L^2(\mathbb{C})$, then

$$(\mathcal{F}f_x)(\zeta) = 2\pi i\, \xi\, (\mathcal{F}f)(\zeta), \qquad (\mathcal{F}f_y)(\zeta) = 2\pi i\, \eta\, (\mathcal{F}f)(\zeta).$$

In complex notation of (4.5), the formulæ above become

$$(\mathcal{F}f_z)(\zeta) = 2\pi i\, \bar{\zeta}\, (\mathcal{F}f)(\zeta), \qquad (\mathcal{F}f_{\bar{z}})(\zeta) = 2\pi i\, \zeta\, (\mathcal{F}f)(\zeta). \tag{6.6}$$

The *Beurling transform* $\mathcal{S}\colon L^2(\mathbb{C}) \to L^2(\mathbb{C})$ is defined as multiplication by $\bar{\zeta}/\zeta$ conjugated by the Fourier transform:

$$\mathcal{F}(\mathcal{S}\omega) = \frac{\bar{\zeta}}{\zeta}\, \mathcal{F}\omega, \qquad \omega \in L^2(\mathbb{C}). \tag{6.7}$$

The Fourier transform preserves the L^2 norm (see Rudin); so does multiplication by $\bar{\zeta}/\zeta$. Therefore identity (6.7) defines $\mathcal{S}$ as an isometry on L^2:

$$\|\mathcal{S}\omega\|_{L^2} = \|\omega\|_{L^2} \tag{6.8}$$

Assuming that both distributional derivatives f_z and $f_{\bar{z}}$ are globally square-integrable, (6.7) in combination with (6.6) reveals that

$$\mathcal{S} \circ \bar{\partial} = \partial \qquad \text{on the space } \dot{W}^{1,2}(\mathbb{C}), \tag{6.9}$$

where

$$\dot{W}^{1,2}(\mathbb{C}) = \{f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})\colon \nabla f \in L^2(\mathbb{C})\} \tag{6.10}$$

is the *Dirichlet space*. Informally speaking, the operator $\mathcal{S}$ 'intertwines' the Wirtinger derivatives ∂ and $\bar{\partial}$; herein lies its main value for solving Beltrami's equation.

6.8 Integral formula for the Beurling transform. Let us now derive an explicit formula for $\mathcal{S}$. To do so, we first observe that (6.5) and (6.9) imply that

$$\mathcal{S} = \partial \circ \mathcal{C} \qquad \text{on the space } C^1_\circ(\mathbb{C}).^3 \tag{6.11}$$

The Cauchy transform commutes with ∂ on $C^1_\circ$, so we let $\phi \in C^1_\circ(\mathbb{C})$ and write

$$(\mathcal{S}\phi)(z) = (\mathcal{C}\phi)_z(z) = -\frac{1}{\pi} \lim_{r\to 0} \int_{|\zeta - z|>r} \frac{\phi_z(\zeta)}{\zeta - z}\, |d\zeta|^2.$$

[3] The identity $\mathcal{S} = \partial \circ \mathcal{C}$ can in fact be taken as *the* definition of $\mathcal{S}$.

The integrand above can be written using the chain rule as

$$\frac{\phi_z(\zeta)}{\zeta - z} = \frac{\phi(\zeta)}{(\zeta - z)^2} + \left(\frac{\phi(\zeta)}{\zeta - z}\right)_z,$$

and applying Green's formula (4.8) to the term $\left(\frac{\phi(\zeta)}{\zeta - z}\right)_z$ shows that

$$(\mathcal{S}\phi)(z) = -\frac{1}{\pi}\lim_{r\to 0}\int_{|\zeta - z|>r} \frac{\phi(\zeta)}{(\zeta - z)^2}\,|d\zeta|^2 - \frac{i}{2\pi}\lim_{r\to 0}\int_{|\zeta - z|=r} \frac{\phi(\zeta)}{\zeta - z}\,d\bar{\zeta}.$$

The second limit on the right vanishes. Indeed,

$$\begin{aligned}\int_{|\zeta - z|=r} \frac{\phi(\zeta)}{\zeta - z}\,d\bar{\zeta} &= \int_{|\zeta - z|=r} \frac{\phi(\zeta) - \phi(z)}{\zeta - z}\,d\bar{\zeta} + \phi(z)\int_{|\zeta - z|=r} \frac{1}{\zeta - z}\,d\bar{\zeta}\\ &\leqslant 2\pi r\cdot 2\|\phi\|_{\mathrm{Lip}(\mathbb{C}\setminus\mathbb{D}_r(z))} + \phi(z)\int_0^{2\pi} \frac{-ie^{-i\theta}}{re^{i\theta}}\,d\theta,\end{aligned}$$

which tends to 0 with r. Here, in the last line, $\|\phi\|_{\mathrm{Lip}(\Omega)}$ denotes the Lipschitz norm of ϕ on Ω, which coincides with $\sup_{z\in\Omega}\|D^{+}\phi(z)\|$, or, simply, the 1–Hölder norm. We conclude that on the space of continuously differentiable functions with compact support, *the Beurling transform is given by the integral*

$$(\mathcal{S}\phi)(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\phi(\zeta)}{(\zeta - z)^2}\,|d\zeta|^2, \qquad \phi\in C^1_\circ(\mathbb{C}), \tag{6.12}$$

which is to be understood in the Cauchy principal value sense.

We initially defined the Beurling transform as the unique isometry of $L^2(\mathbb{C})$ satisfying (6.7). Let us now verify that the singular integral formula (6.12) extends to square integrable functions. While it is possible to infer this convergence property by studying the so-called maximal transforms (analogous to maximal functions), here we provide an elementary argument due to Mateu and Verdera [MV06, Proof of Theorem 3].

First, we find the Beurling transform of the characteristic function of the disc $\mathbb{D}_r(a)$. In the special case when a is the origin and $r = 1$, the function

$$h(z) = \begin{cases} \bar{z}, & |z|\leqslant 1,\\ z^{-1}, & |z|>1\end{cases}$$

satisfies $h_{\bar{z}} = \chi_{\mathbb{D}}$ and belongs to the space $W^{1,2}_{\diamond}(\mathbb{C})$ of locally integrable functions with square integrable distributional derivatives. From the property

(6.9) we infer that

$$(\mathcal{S}\chi_{\mathbb{D}})(z) = -\frac{1}{z^2}\,\chi_{\mathbb{C}\setminus\mathbb{D}}(z).$$

For the general disc $\mathbb{D}_r(a)$, we change the variables to obtain

$$(\mathcal{S}\chi_{\mathbb{D}_r(a)})(z) = -\frac{r^2}{(z-a)^2}\,\chi_{\mathbb{C}\setminus\mathbb{D}_r(a)}(z). \tag{6.13}$$

We next observe that $\mathcal{S}$ is symmetric with respect to the inner product on L^2. Indeed, if f and g are smooth and compactly supported, then

$$\begin{aligned}\int_{\mathbb{C}} f(\zeta)\,(\mathcal{S}g)(\zeta)\,|d\zeta|^2 &= -\frac{1}{\pi}\lim_{\epsilon\to 0}\iint_{|\tau-\zeta|>\epsilon}\frac{f(\zeta)\,g(\tau)}{(\tau-\zeta)^2}\,|d\tau|^2\,|d\zeta|^2\\ &= \int_{\mathbb{C}}(\mathcal{S}f)(\tau)\,g(\tau)\,|d\tau|^2,\end{aligned} \tag{6.14}$$

and by density of $C_\circ^\infty$ in L^2, (6.14) extends to all square integrable f and g.

Finally, assuming $f \in L^2(\mathbb{C})$, we use the results of (6.13) and (6.14) to write

$$\begin{aligned}-\frac{1}{\pi}\int_{|\zeta-z|>r}\frac{f(\zeta)}{(\zeta-z)^2}\,|d\zeta|^2 &= \frac{1}{\pi r^2}\int_{\mathbb{C}}(\mathcal{S}\chi_{\mathbb{D}_r(z)})(\zeta)\,f(\zeta)\,|d\zeta|^2\\ &= \frac{1}{|\mathbb{D}_r(z)|}\int_{\mathbb{D}_r(z)}(\mathcal{S}f)(\tau)\,|d\tau|^2.\end{aligned}$$

The integral on the right is well-defined because $\mathcal{S}$ is an isometry on L^2, whereby $\mathcal{S}f \in L^2(\mathbb{C})$. By virtue of Lebesgue's Differentiation theorem, as we let $r \to 0$, the quantity on the right attains in the limit the finite value $(\mathcal{S}f)(z)$ at almost every $z \in \mathbb{C}$, and so the principal value of the integral on the left exists almost everywhere.

Later in §6.16, we will briefly discuss the deep fact that $\mathcal{S}$ extends to a continuous operator on L^p beyond mere $p = 2$. It is noteworthy that the above argument of Mateu and Verdera applies there as well. That is to say, the integral in formula (6.12) exists almost everywhere in the principal value sense also for $\phi \in L^p(\mathbb{C})$ with $2 \leqslant p < \infty$.

6.9 Cauchy transform of a mapping with compact support. The theory of the Cauchy transform is particularly nice for functions of compact support. In this section, we follow [AIM08].

Let us note that if the compactly supported ϕ is merely integrable, then, on the strength of Fubini's Theorem, we have

$$\int_F |(\mathcal{C}\phi)(z)|\,|dz|^2 \leqslant -\frac{1}{\pi}\lim_{\epsilon\to 0}\iint_{\{\zeta, z\in F\colon\,|\zeta-z|>\epsilon\}} \left|\frac{\phi(\zeta)}{\zeta - z}\right|\,|d\zeta|^2\,|dz|^2 < \infty$$

for every compact set $F \subset \mathbb{C}$. This means that *if $\phi \in L^1_\circ$ then the function $\mathcal{C}\phi$ belongs to $L^1_{\mathrm{loc}}(\mathbb{C})$, and the principal value integral in* (6.3) *converges at almost every point $z \in \mathbb{C}$.*

Now let $1 < p < \infty$ and let f be a p–integrable compactly supported complex function. Fix a sufficiently large radius R so that f vanishes outside the disc $\mathbb{D}_R$. Under this hypothesis, the convolution with the *truncated Cauchy kernel*

$$\mathcal{K}_R(\tau) = -\frac{1}{\pi\tau}\,\chi_{\mathbb{D}_{3R}}(\tau) \tag{6.15}$$

can replace the Cauchy transform of f inside the disc of radius $2R$. Indeed, if $|z| < 2R$ then

$$(\mathcal{K}_R * f)(z) = \int_{\mathbb{C}} \mathcal{K}_R(\zeta - z)\,f(z)\,|d\zeta|^2 = -\frac{1}{\pi}\int_{|\zeta - z|<3R} \frac{f(\zeta)}{\zeta - z}\,|d\zeta|^2 = (\mathcal{C}f)(z).$$

Young's inequality for convolutions then implies

$$\begin{aligned}\|\mathcal{C}f\|_{L^p(\mathbb{D}_{2R})} &\leqslant \|\mathcal{K}_R * f\|_{L^p}\\ &\leqslant \|\mathcal{K}_R\|_{L^1}\|f\|_{L^p} = 6R\,\|f\|_{L^p(\mathbb{C})}.\end{aligned}$$

This shows that $\mathcal{C}\colon L^p_\circ \to L^p_{\mathrm{loc}}$ and that $(\mathcal{C}f)(z)$ exists in the principal value sense for almost all $z \in \mathbb{D}_R$.

Most importantly, the Cauchy transform of a compactly supported function is analytic near ∞. Indeed, for $|\zeta| < R$ and $|z| > R$, using the geometric series, we can write

$$-\frac{1}{\zeta - z} = \frac{1}{z\left(1 - \frac{\zeta}{z}\right)} = \frac{1}{z}\left(1 + \frac{\zeta}{z} + \frac{\zeta^2}{z^2} + \cdots\right).$$

Since the convergence of partial sums on the right is uniform whenever the modulus of ζ/z is below 1, it follows that $\mathcal{C}f$ is analytic with the Laurent series expansion

$$(\mathcal{C}f)(z) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{z^k}\left(\int_{\mathbb{C}} \zeta^{k-1}\,f(\zeta)\,|d\zeta|^2\right)$$

near infinity. Hence, the principal value integral $(\mathcal{C}f)(z)$ converges at every

$z \in \mathbb{C} \setminus \mathbb{D}_R$.

These observations justify why identities (6.5) and (6.11) extend to $L^p_\circ$ (in the distributional sense). We combine all of this in the following theorem, but only in the case $p = 2$ for now; later, we will extend Theorem 6.10 to the case $p \geqslant 2$ — with the help of the results listed in 6.16.

6.10 Theorem. *If $f \in L^2_\circ(\mathbb{C})$ has compact support contained in a disc D, then $\mathcal{C}f$ belongs to $L^2_{\mathrm{loc}}(\mathbb{C})$, is analytic outside D, and is, therefore, well-defined for almost every $z \in D$ and for every $z \in \mathbb{C} \setminus D$. Further, we have the identities*

$$\bar\partial \circ \mathcal{C} = \mathrm{id}, \quad \partial \circ \mathcal{C} = \mathcal{S} \qquad \text{on the space } L^2_\circ(\mathbb{C}). \tag{6.16}$$

Consequently, the Cauchy transform is an operator $\mathcal{C}\colon L^2_\circ(\mathbb{C}) \to \dot W^{1,2}(\mathbb{C})$.

PROOF The first claim of the theorem was proved in 6.9 above.

Let $\varphi \in C^\infty_\circ$ be a smooth test function. As $\mathcal{C}f$ is well-defined almost everywhere in $\mathbb{C}$, we can compute:

$$\int_{\mathbb{C}} \varphi\, (\mathcal{C}f)_{\bar z} = -\int_{\mathbb{C}} \varphi_{\bar z}\, \mathcal{C}f = \int_{\mathbb{C}} \mathcal{C}\varphi_{\bar z}\, f = \int_{\mathbb{C}} \varphi\, f,$$
$$\int_{\mathbb{C}} \varphi\, (\mathcal{C}f)_{z} = -\int_{\mathbb{C}} \varphi_{z}\, \mathcal{C}f = \int_{\mathbb{C}} \mathcal{C}\varphi_{z}\, f = \int_{\mathbb{C}} \mathcal{S}\varphi\, f = \int_{\mathbb{C}} \varphi\, \mathcal{S}f,$$

where in the second line we use (6.11) and (6.14). This shows that the identities

$$(\mathcal{C}f)_{\bar z} = f \qquad \text{and} \qquad (\mathcal{C}f)_z = \mathcal{S}f$$

hold in the sense of distributions, thus proving (6.16).

As a consequence,

$$\big\|(\mathcal{C}f)_{\bar z}\big\|_{L^2} = \|f\|_{L^2} < \infty,$$
$$\|(\mathcal{C}f)_z\|_{L^2} = \|\mathcal{S}f\|_{L^2} = \|f\|_{L^2} < \infty.$$

We now see that $\mathcal{C}f \in \dot W^{1,2}(\mathbb{C})$, where $\dot W^{1,2}(\mathbb{C})$ is defined by (6.10). $\square$ THEOREM 6.10

Compactly Supported Dilatation

In the following few sections, we establish, using identities (6.8), (6.9), and Theorem 6.10, that the Beltrami equation is solvable in $W^{1,2}_{\mathrm{loc}}$; moreover, we furnish integral representation formulæ for Sobolev solutions.

6.11 Theorem. *If μ is a compactly supported measurable function satisfying $\|\mu\|_{L^\infty(\mathbb{C})} < 1$, then the Beltrami equation $f_{\bar{z}} = \mu f_z$ has the unique $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ solution of the form*

$$f^\mu(z) = z + (\mathcal{C}g^\mu)(z), \tag{6.17}$$

where

$$g^\mu = \mu + (\mu\mathcal{S})\mu + (\mu\mathcal{S} \circ \mu\mathcal{S})\mu + (\mu\mathcal{S} \circ \mu\mathcal{S} \circ \mu\mathcal{S})\mu + \cdots . \tag{6.18}$$

6.12 Principal solutions. We will now reverse-engineer the Beltrami equation to find the most natural class of solutions.

Let $f: \mathbb{C} \to \mathbb{C}$ be a quasiconformal mapping which satisfies $f_{\bar{z}} = \mu f_z$ for some compactly supported μ with $\|\mu\|_{L^\infty} < 1$.

We may extend μ to the Riemann sphere $\overline{\mathbb{C}}$ by setting $\mu(\infty) = 0$. Similarly, f extends to a homeomorphism of $\overline{\mathbb{C}}$ with $f(\infty) = \infty$ as it is initially assumed to be a homeomorphism of $\mathbb{C}$ onto itself.

The hypothesis that $f_{\bar{z}} = 0$ outside a compact set has two implications. First and foremost, the Wirtinger derivative $f_{\bar{z}}$ is automatically square-integrable over $\mathbb{C}$. Second, by Weyl's Lemma 4.12, f is analytic in a neighbourhood of infinity. Therefore, f admits the Laurent series expansion

$$f(z) = b_1 z + b_0 + \sum_{k=1}^{\infty} \frac{b_{-k}}{z^k}$$

near ∞ (cf. §4.3). The uniqueness Corollary 6.5 says that f is determined by the condition $f_{\bar{z}} = \mu f_z$ uniquely up to post-composition with a similarity. Thus, we can normalize f so that $b_0 = 0$ and $b_1 = 1$.[4] Let us appoint the symbol f^μ to the (unique) quasiconformal mapping of $\mathbb{C}$ whose complex dilatation is $\mu \in L^\infty_\circ$ and which expands as

$$f^\mu(z) = z + \sum_{k=1}^{\infty} \frac{a_k}{z^k} \tag{6.19}$$

in a neighbourhood of ∞.[5] Note that $f^\mu = \phi \circ f$ inside the support of μ, where ϕ is a conformal on the support of μ.

The normalization (6.19) entails that the mapping g^μ defined by

$$g^\mu(z) = f^\mu(z) - z$$

[4] Recall from elementary complex analysis: a Möbius trasformation that fixes the point ∞ is necessarily a similarity of $\mathbb{C}$.

[5] In other words, we require $\lim_{z\to\infty}(f^\mu(z) - z) = 0$.

has both Wirtinger derivatives in $L^2(\mathbb{C})$, as the computations

$$g_z^\mu = f_z^\mu - 1, \qquad g_{\bar z}^\mu = f_{\bar z}^\mu \tag{6.20}$$

show. A direct substitution of the formulæ (6.20) into the Beltrami equation yields the *auxiliary equation*[6]

$$g_{\bar z}^\mu = \mu\, g_z^\mu + \mu. \tag{6.21}$$

The definition of g^μ renders it absolutely continuous on lines and thus forces it to be of the class $\dot W^{1,2}(\mathbb{C})$ (cf. (6.10)). This makes it possible to apply identity (6.9) to the $\bar\partial$ derivative of g^μ in (6.21). We conclude that the homeomorphic solution $f^\mu = z + g^\mu$ of the Beltrami equation satisfies the normalization condition (6.19) only if

$$g_z^\mu = \mu\, \mathcal{S} g_{\bar z}^\mu + \mu. \tag{6.22}$$

In the next section, we shall obtain the formulæ for the Sobolev solutions of the Beltrami equation with compactly supported dilatation by studying the singular integral equation (6.22).

6.13 Integral representation formulæ. In view of the preceding discussion, we are after the solution of the form

$$f^\mu(z) = z + g^\mu(z) \qquad \text{with} \qquad g^\mu \in \mathcal{O}\left({}^1\!/\!{}_{|z|}\right) \text{ near } \infty. \tag{6.23}$$

We shall henceforth call such $f^\mu \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ *the principal solution.*

We write the singular integral equation (6.22) in the form

$$(\mathrm{id} - \mu\mathcal{S})\, g_{\bar z}^\mu = \mu. \tag{6.24}$$

The Beurling transform $\mathcal{S}$ is an isometry on L^2, and thus $\|\mu\mathcal{S}\| \leqslant \|\mu\|_\infty < 1$. Therefore, the inverse of the integral operator $(\mathrm{id} - \mu\mathcal{S})$ is given by the convergent Neumann (geometric) series

$$(\mathrm{id} - \mu\mathcal{S})^{-1} = \mathrm{id} + \mu\mathcal{S} + \mu\mathcal{S} \circ \mu\mathcal{S} + \mu\mathcal{S} \circ \mu\mathcal{S} \circ \mu\mathcal{S} + \cdots,$$

and equation (6.24) is uniquely solvable for $g_{\bar z}^\mu$ in $L^2(\mathbb{C})$. Dilatation μ has

[6] Observe that equation (6.21) is in fact an *inhomogeneous Beltrami equation.* We will encounter more auxiliary equations of this type later in .. .

compact support and so does

$$g^\mu_{\bar z} = (\mathrm{id} - \mu\mathcal{S})^{-1}\mu.$$

By Theorem 6.10, the Cauchy transform of $g^\mu_{\bar z}$ is well-defined almost everywhere in $\mathbb{C}$, is analytic outside a compact set, and possesses the decay of order $\mathcal{O}(1/|z|)$ at infinity. We can thus use the Cauchy transform to recover $g \in \dot W^{1,2}(\mathbb{C})$ and construct the principal solution to the Beltrami equation. We have proved Theorem 6.11.

Unfortunately, the L^2 theory of the Cauchy transform is not enough to conclude that the principal solution furnished by formulæ (6.17)–(6.18) is a homeomorphism. As a matter of fact, the Cauchy transform of an L^2 function need not be even continuous, and higher integrability of g^μ is required.

L^p Theory for Operators $\mathcal{C}$ and $\mathcal{S}$

We still work under the hypothesis that the dilatation μ vanishes outside of a bounded set. We need to establish that solutions given by Theorem 6.11 are quasiconformal homeomorphisms of $\mathbb{C}$.

6.14 Re-defining the Cauchy transform. First, let us give meaning to the Cauchy trasform of a mapping whose support is not compact.

In 6.9, we saw that whenever $2 < p < \infty$ and the function $f \in L^p(\mathbb{C})$ vanishes outside a compact set, the principal value integral

$$(\mathcal{C}f)(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z}\,|d\zeta|^2$$

exists almost everywhere in $\mathbb{C}$. However if f is not compactly supported, we cannot guarantee that the integral exists, since the Cauchy kernel $\zeta \mapsto -1/\pi\zeta$ is not in $L^q(\mathbb{C})$ for $1 < q < 2$. This issue is resolved by re-defining

$$(\mathcal{C}f)(z) = -\frac{1}{\pi}\int_{\mathbb{C}} f(\zeta)\left[\frac{1}{\zeta - z} - \frac{1}{\zeta}\right]|d\zeta|^2, \qquad f \in L^p(\mathbb{C}),\ 2 < p < \infty, \tag{6.25}$$

where we interpret the singular integral in the principal value sense, as before. As a function of ζ, the factor

$$\frac{1}{\zeta - z} - \frac{1}{\zeta} = \frac{z}{\zeta(\zeta - z)}$$

belongs to all $L^q(\mathbb{C})$ with $1 < q < 2$, and Hölder's inequality entails

$$|(\mathcal{C}f)(z)| \leqslant \frac{1}{\pi}\|f\|_{L^p}\left\|\frac{z}{\zeta(\zeta-z)}\right\|_{L^q} < \infty$$

for q satisfying $1/p + 1/q = 1$. This ensures that the principal value integral in (6.25) is well-defined almost everywhere in $\mathbb{C}$.

For a compactly supported function f, the formula (6.25) defines a function which differs from the original Cauchy transform of f by an additive constant: $(\mathcal{C}f)(z) - (\mathcal{C}f)(0)$. Hence the identity $(\mathcal{C}f)_{\bar{z}} = f$ still holds with this new definition. As compactly supported continuous functions are dense in L^p for $2 < p < \infty$, the operator defined by (6.25) possesses the crucial property $\bar{\partial} \circ \mathcal{C} = \mathrm{id}$ on L^p as well.

We remark that in the case $p = 2$, the integral (6.25) does not make sense either. It is still possible to obtain the identity $\bar{\partial} \circ \mathcal{C} = \mathrm{id}$ on L^2 by altering the definition yet again:

$$(\mathcal{C}f)(z) = -\frac{1}{\pi}\int_{\mathbb{C}} f(\zeta)\left[\frac{1}{\zeta - z} - \frac{\chi_{\mathbb{C}\setminus\mathbb{D}}(\zeta)}{\zeta}\right]|d\zeta|^2, \qquad f \in L^2(\mathbb{C}).$$

As before, $\mathcal{C}f$ is interpreted as the equivalence class of functions modulo an additive constant. For details, see [AIM08, Section 4.3.2].

6.15 Theorem (Continuity of the Cauchy transform). *Let $f \in L^p(\mathbb{C})$ for $2 < p < \infty$. The mapping $\mathcal{C}f$ defined by (6.25) is uniformly Hölder continuous with exponent $1 - \frac{2}{p}$.*

PROOF We follow [Ahl06] in this proof. Let q be the Hölder conjugate exponent to p, $1/p + 1/q = 1$. When $z \neq 0$, a change of variables $\zeta = \tau z$ shows that

$$\int_{\mathbb{C}}\left|\frac{1}{\zeta(\zeta-z)}\right|^q |d\zeta|^2 = |z|^{-2(q-1)}\int_{\mathbb{C}}\left|\frac{1}{\tau(\tau-1)}\right|^q |d\tau|^2,$$

and hence

$$\left\|\frac{z}{\zeta(\zeta-z)}\right\|_{L^q} = |z|^{1-\frac{2(q-1)}{q}}\left\|\frac{1}{\tau(\tau-1)}\right\|_{L^q}.$$

Then, by Hölder's inequalty,

$$|(\mathcal{C}f)(z)| \leqslant C_p\,\|f\|_{L^p}\,|z|^{1-\frac{2}{p}}, \tag{6.26}$$

where the constant

$$C_p = \left\| \frac{1}{\tau(\tau-1)} \right\|_{L^q}$$

depends only on p. As $(\mathcal{C}f)(0) = 0$ by definition, the inequality (6.26) holds trivially at $z = 0$.

Defining $\tilde{f}(\zeta) = f(\zeta + z_1)$, we have

$$\begin{aligned}(\mathcal{C}\tilde{f})(z_2 - z_1) &= -\frac{1}{\pi}\int_{\mathbb{C}} f(\zeta + z_1)\left[\frac{1}{\zeta + z_1 - z_2} - \frac{1}{\zeta}\right] |d\zeta|^2 \\ &= -\frac{1}{\pi}\int_{\mathbb{C}} f(\tau)\left[\frac{1}{\tau - z_2} - \frac{1}{\tau - z_1}\right] |d\tau|^2 \\ &= (\mathcal{C}f)(z_2) - (\mathcal{C}f)(z_1),\end{aligned}$$

where we changed variables $\zeta = \tau - z_1$. Applying (6.26) to $(\mathcal{C}\tilde{f})(z_2 - z_1)$ yields

$$|(\mathcal{C}f)(z_2) - (\mathcal{C}f)(z_1)| \leqslant C_p \, \|f\|_{l^p} \, |z_2 - z_1|^{1-\frac{2}{p}}, \tag{6.27}$$

and the claim follows. □ THEOREM 6.15

6.16 **L^p theory for the Beurling transform.** Now that we wish to look for solutions that possess higher integrability, we need to extend the Beurling transform to L^p with $2 < p < \infty$, while making sure that the integral equation (6.24) is still solvable by the Neumann iteration in L^p. The latter is fulfilled if $\|\mu\mathcal{S}\| < 1$, so we require that $k\|\mathcal{S}\|_{L^p\to L^p} < 1$, where $k = \|\mu\|_{L^\infty}$. In other words, in addition to being an isometry on L^2, $\mathcal{S}$ should not perturb the norms too much as an operator acting on the space L^p. This is asserted by the theorem due to Calderón and Zygmund ([Ahl06, Chapter V], [SM93]).

The Beurling transform is a bounded operator on $L^p(\mathbb{C})$ for every $p > 1$.

Further, the Riesz–Thorin convexity theorem states that the operator norm $\ln(\|\mathcal{S}\|_{L^p\to L^p})$ is a convex function of $\frac{1}{p}$ ([Ahl06, Chapter V]). Convexity implies, in particular, the following.

The operator norm $\|\mathcal{S}\|_{L^p\to L^p}$ is a continuous function of p with

$$\lim_{p\to 2} \|\mathcal{S}\|_{L^p\to L^p} = 1.$$

We collect this information if the following proposition.

6.17 Proposition. *For every $k \in (0,1)$ there is $p \in (2,\infty)$ such that $\|\mathcal{S}\|_p < \frac{1}{k}$.*

Now that we know that the Beurling transform extends continuously beyond L^2, we can apply the same reasoning as in §§6.8–6.10 to conclude the following.

6.18 Proposition. *Assume $2 < p < \infty$.*

(i) *The integral in formula (6.12) for the Beurling transform exists almost everywhere for $\phi \in L^p(\mathbb{C})$.*

(ii) *The Cauchy transform is an operator $\mathcal{C}\colon L^p_\circ(\mathbb{C}) \to \dot{W}^{1,p}(\mathbb{C}) \cap C^\alpha(\mathbb{C})$, with Hölder exponent $\alpha = 1 - {}^2\!/_p$. Moreover, the Cauchy transform of an $L^p_\circ$ function is analytic near ∞ with decay of order $\mathcal{O}({}^1\!/_z)$.*

(iii) *Identities (6.16) hold in the distributional sense on the space $L^p(\mathbb{C})$.*

Homeomorphic Solutions

We use the idea from [Ahl06] to make use of Lemma 6.19 below to establish a condition on μ so that the principal solution of (BE) is a global homeomorphism. Hereafter, we let $k = \|\mu\|_{L^\infty} < 1$ and fix $p > 2$ given by Proposition 6.17.

6.19 Lemma (A generalization of Weyl's Lemma). *If α and β are continuous maps whose distributional derivatives are in $L^1_{\text{loc}}(\Omega)$ and satisfy $\alpha_{\bar z} = \beta_z$, then there exists a function $f \in C^1(\Omega)$ which satisfies $f_z = \alpha$ and $f_{\bar z} = \beta$.*

This lemma provides a criterion under which the system of differential equations

$$\begin{cases} f_z = \alpha \\ f_{\bar z} = \beta \end{cases}$$

admits a continuously differentiable solution f.

PROOF It may be useful to recall some basics from multivariate calculus before getting to the proof. We say that a differential form ω is *exact* if there is a continuously differentiable potential function f such that $\omega = df$.

Comparing the expressions $\omega = \alpha\, dz + \beta\, d\bar{z}$ and $df = f_z\, dz + f_{\bar{z}}\, d\bar{z}$ we see that $\omega = df$ is equivalent to saying that $f_z = \alpha$ and $f_{\bar{z}} = \beta$. On the other hand, ω is exact if and only if the line integral of ω is path-independent, or, equivalently, the integral of ω against every closed curve vanishes. In view of Green's theorem for C^1 functions, this is so if and only if $(f_z)_{\bar{z}} = (f_{\bar{z}})_z$.

Fix an arbitrary subdomain $\Omega' \subset\subset \Omega$ with a piecewise C^1 boundary (this suffices for our purposes). By the process of mollification we can find approximating families of smooth functions α_ϵ and β_ϵ which converge to α and β uniformly on Ω' as $\epsilon \to 0$. Since convolution commutes with derivatives, $\bar{\partial}\alpha_\epsilon = \partial\beta_\epsilon$ for every positive ϵ. Using Green's formula (4.7), we see that

$$\int_{\partial\Omega'} \alpha_\epsilon(\zeta)\, d\zeta + \beta_\epsilon(\zeta)\, d\bar{\zeta} = \int_{\Omega'} \big(\partial\beta_\epsilon(\zeta) - \bar{\partial}\alpha_\epsilon(\zeta)\big)\; d\zeta \wedge d\bar{\zeta} = 0.$$

The integral on the left tends to $\int_{\partial\Omega'} \alpha\, dz + \beta\, d\bar{z}$ as we let $\epsilon \to 0$. The integral on the right vanishes if $\alpha_{\bar{z}} = \beta_z$. This confirms the claim. □ LEMMA 6.19

6.20 Theorem. *Let $\mu \in L^\infty_\circ(\Omega) \cap C(\Omega)$ and fix $p > 2$ so that $k\|\mathcal{S}\|_p < 1$. Under the additional hypothesis that the distributional derivative μ_z lies in L^p, the principal solution to the Beltrami equation $f_{\bar{z}} = \mu\, f_z$ is continuously differentiable.*

PROOF Suppose a function $\tilde{f}$ satisfies $\tilde{f}_{\bar{z}} = \mu\tilde{f}_z$ and set $\lambda = \tilde{f}_z$. By Lemma 6.19, $\tilde{f}$ is continuously differentiable if λ is continuous and $\lambda_{\bar{z}} = (\mu\lambda)_z$ in the distributional sense. Expanding the latter, we require that $\lambda_{\bar{z}} = \mu\lambda_z + \mu_z\lambda$. Setting $\sigma = \ln\lambda$, we see that this will be so if $\sigma \in C(\Omega)$ is such that

$$\sigma_{\bar{z}} = \mu\,\sigma_z + \mu_z. \tag{6.28}$$

We are thus led to study the solutions of the integral equation

$$(\mathrm{id} - \mu\mathcal{S})\,\sigma_{\bar{z}} = \mu_z \tag{6.29}$$

where $\sigma_{\bar{z}}$ is the unknown.

By the hypothesis, $\mu_z \in L^p_\circ$ and $\|\mu\mathcal{S}\|_p \leqslant k\|\mathcal{S}\|_p < 1$ whence equation (6.29) is solved by $\sigma_{\bar{z}} = (\mathrm{id} - \mu\mathcal{S})^{-1}\mu_z \in L^p$ whose support is compact.

It follows that $\sigma = \mathcal{C}\sigma_{\bar{z}}$ is continuous, by Proposition 6.18 (*ii*). Hence $\lambda = e^\sigma$ is continuous. Since $(e^\sigma)_{\bar{z}} = (\mu e^\sigma)_z$, Lemma 6.19 guarantees the existence of a continuously differentiable $\tilde{f}$ satisfying

$$\tilde{f}_z = e^{\sigma} \qquad \text{and} \qquad \tilde{f}_{\bar{z}} = \mu e^{\sigma}. \tag{6.30}$$

Further, as $\sigma_{\bar{z}}$ has compact support, $\sigma = \mathcal{C}\sigma_{\bar{z}}$ (defined up to an additive constant) is analytic near infinity, and we may choose the constant so that $\sigma(z) \in \mathcal{O}(1/z)$ near infinity. Then $\tilde{f}_z$ is analytic near ∞, and $\lim_{z\to\infty} \tilde{f}_z = 1$. Consequently, $\tilde{f}_z - 1$ belongs to $L^p(\mathbb{C})$. We can therefore calculate

$$\tilde{f}_z - 1 = \mathcal{C}((\tilde{f}_z - 1)_{\bar{z}}) = \mathcal{C}((\tilde{f}_{\bar{z}})_z) = \mathcal{S}\tilde{f}_{\bar{z}},$$

in view of Proposition 6.18 (iii). Because $\mathcal{S}$ is continuous as an operator on L^p (Proposition 6.17), we see that $\tilde{f}_{\bar{z}} \in L^p(\mathbb{C})$.

We now have gathered enough information to conclude that $\tilde{f}$ is the principal solution of the Beltrami equation $f_{\bar{z}} = \mu\, f_z$. The uniqueness of the principal solution now implies that $f^{\mu} = \tilde{f}$.

It is noteworthy that f^{μ} admits the integral representation

$$f^{\mu}(z) = z + (\mathcal{C}\mu e^{\sigma})(z), \tag{6.31}$$

where

$$\sigma = \mathcal{C}\big(\mu_z + (\mu\mathcal{S})\mu_z + (\mu\mathcal{S} \circ \mu\mathcal{S})\mu_z + (\mu\mathcal{S} \circ \mu\mathcal{S} \circ \mu\mathcal{S})\mu_z + \cdots\big)$$

up to an additive constant, see [AIM08, Theorem 5.2.3]. □ THEOREM 6.20

6.21 Remark on auxiliary equations. On two occasions above, we encountered the inhomogeneous Beltrami equation

$$\sigma_{\bar{z}} = \mu\,\sigma_z + \varrho \tag{6.32}$$

with different source terms ϱ. In 6.13, assuming merely that $f \in W^{1,2}_{\mathrm{loc}}$ lead, by virtue of Weyl's lemma, to the auxiliary equation (6.21) with $\varrho = \mu$. Similarly, when we sought for a continuously differentiable f in § 6.20, we obtaind (6.28), in which $\varrho = \mu_z$.

Inhomogeneous Beltrami equations of the type (6.32) are a useful tool for studying the regularity of the solutions to the actual Beltrami equation (BE). As a matter of fact, the solutions to (BE) can be as smooth as μ and ϱ in (6.32) allow. For a detailed exposition of this approach, see Sections 5.1 and 5.2 in [AIM08].

6.22 Homeomorphicity of solutions. We can deduce local injectivity of the C^1 principal solution immediately from the formulæ (6.30) and (6.31). Indeed, the Jacobian of f^μ (cf. (4.21)) is strictly positive everywhere:

$$Jf = \left|f_z\right|^2 - \left|f_{\bar{z}}\right|^2 = \left|e^{2\sigma}\right|\left(1 - |\mu|^2\right) > 0.$$

The inverse function theorem thus implies that f is a local homeomorphism in the finite plane $\mathbb{C}$.

The condition $\lim_{z\to\infty} f(z) = \infty$ allows us to extend f the Riemann sphere $\overline{\mathbb{C}}$ by defining $f(\infty) = \infty$. Recall that the principal solution has the Laurent series expansion $z + \mathcal{O}(1/z)$ near infinity. Congugating f^μ with inversion produces the mapping $h(z) = 1/f^\mu(1/z)$, and $h(z) \in \mathcal{O}(z)$ near zero. We also see that $h'(0) = 1$ (cf. § 4.3). This implies that h is a homeomorphism in a neighbourhood of the origin; consequently, f must be homeomorphic in a neighbourhood of infinity. We can therefore conclude that f is a local homeomorphism of the Riemann sphere $\overline{\mathbb{C}}$.

Lastly, on the strength of the monodromy theorem and the fact that $\overline{\mathbb{C}}$ is simply connected, f must be a global homeomorphism of $\overline{\mathbb{C}}$.

We gather our findings in the following theorem.

6.23 Theorem. *If $\mu \in C_\circ^\infty(\mathbb{C})$ is such that $\|\mu\|_{L^\infty} = k < 1$, then the principal solution to the Beltrami equation $f_{\bar{z}} = \mu\, f_z$ is a C^1 homeomorphism of the Riemann sphere $\overline{\mathbb{C}}$ and is K–quasiconformal with $K = \frac{1+k}{1-k}$.*

PROOF Clearly, a smooth and compactly supported Beltrami coefficient μ satisfies the hypothesis of Theorem 6.20, whereby the principal solution f^μ (which exists by Theorem 6.11) is of class $C^1(\mathbb{C})$. According to § 6.22, f^μ is a global homeomorphism of $\overline{\mathbb{C}}$. Since f^μ fixes the point ∞, it acts as homeomorphically on the finite plane $\mathbb{C}$. In view of § 6.2, f^μ is indeed K–quasiconformal in $\mathbb{C}$. □ THEOREM 6.23

Relaxing the Assumptions on the Dilatation

6.24 Good Approximation Lemma. *Suppose that $\{\mu_n\}_{n\in\mathbb{N}}$ is a sequence of Beltrami quotients in $L^\infty(\mathbb{C})$, such that $\|\mu_n\|_{L^\infty} \leqslant k < 1$ for every n and the pointwise limit*

$$\mu(z) = \lim_{n\to\infty} \mu_n(z)$$

exists almost everywhere. For every n, let $f_n\colon \mathbb{C} \to \mathbb{C}$ be the homeomorphic solution to

$$f_{\bar z} = \mu_n \, f_z,$$

normalized so that $f_n(0) = 0$, $f_n(1) = 1$. Under this hypothesis, the limit

$$f(z) = \lim_{n\to\infty} f_n(z)$$

exists at every $z \in \mathbb{C}$, the convergence is uniform on compact sets, and the limit mapping is a $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ homeomorphic solution of the limit equation

$$f_{\bar z} = \mu \, f_z.$$

PROOF By Theorem 5.12, the sequence $\{f_n\}$ is normal, and so there is a subsequence $\{f_{n_k}\}$ which converges uniformy on compact sets to f. By the same token, f is indeed quasiconformal. The limit mapping necessarily fixes 0 and 1, and thus it suffices to show that f solves equation the limit equation, since, by uniqueness of the normalized solution, all convergent subsequences must have the same limit.

We use the quasisymmetry characterization. For every fixed disc $\mathbb{D}_R$, the area formula from the previous chapter entails

$$\int_{\mathbb{D}_R} \left\|Df_n(z)\right\|^2 dz \leqslant K \int_{\mathbb{D}_R} Jf_n(z)\, dz = K\,|f_n(\mathbb{D}_R)| \leqslant K\,\pi\,\eta(R)^2.$$

This shows that the L^2 norms of Df_n are locally uniformly bounded and hence we can extract a subsequence $\{f_{n_j}\} \subset \{f_{n_k}\}$ such that $\partial f_{n_j} \rightharpoonup g$ and $\bar\partial f_{n_j} \rightharpoonup h$ weakly in $L^2_{\mathrm{loc}(\mathbb{C})}$. Further, consider a smooth test function $\varphi \in C^\infty_\circ(\mathbb{C})$. Weak convergence then implies

$$\int_{\mathbb{C}} \varphi_z(\zeta)\, f(\zeta)\, d\zeta = \lim_{j\to\infty} \int_{\mathbb{C}} \varphi_z(\zeta)\, f_{n_j}(\zeta)\, d\zeta = -\int_{\mathbb{C}} \varphi(\zeta)\, g(\zeta)\, d\zeta$$

and hence $g = f_z$ in the sense of distributions. A similar reasoning shows that $h = f_{\bar z}$. Thus f_{n_j} converge weakly in $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ and locally uniformly on $\mathbb{C}$ to f.

Lastly, if R is sufficiently large so that φ is compactly supported in $\mathbb{D}_R$, then

$$\int_{\mathbb{C}} \varphi(\zeta) \left(\bar\partial f_{n_j}(\zeta) - \mu(\zeta)\,\partial f_{n_j}(\zeta)\right) d\zeta = \int_{\mathbb{C}} \varphi(\zeta) \left(\mu_{n_j}(\zeta) - \mu(\zeta)\right) \partial f_{n_j}(\zeta)\, d\zeta, \tag{6.33}$$

because every f_{n_j} solves $f_{\bar z} = \mu_{n_j} f_z$. The term on the left goes to

$$\int_{\mathbb{C}} \varphi(\zeta)\left(f_{\bar z}(\zeta) - \mu(\zeta) f_z(\zeta)\right) d\zeta$$

by weak $L^2(\mathbb{D}_R)$ convergence of the derivatives. The right side is bounded by

$$\|\varphi(\mu_{n_j} - \mu)\|_{L^2} \|Df_{n_j}\|_{L^2(\mathbb{D}_R)} \leqslant \sqrt{\pi K}\eta(R)\|\varphi(\mu_{n_j} - \mu)\|_{L^2}.$$

The Dominated Convergence Theorem of Lebesgue implies that this bound goes to 0 as we let $j \to \infty$. We conclude that f is indeed a solution of the limit equation. □ Lemma 6.24

We conclude this final chapter by completing the proof of the Measurable Riemann Mapping Theorem.

Proof of Theorem 6.3 Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence in $C_\circ^\infty(\mathbb{C})$ of Beltrami quotients that approximate μ,

$$\mu_n \to \mu \qquad \text{pointwise almost everywhere in } C,$$

and that satisfy $\|\mu_n\|_{L^\infty} \leqslant k < 1$ for all n.

By Theorem 6.23, for each $n \in \mathbb{N}$ there is a C^1 quasiconformal homeomorphism f_n of $\mathbb{C}$ which is a principal solution of the Beltrami equation $f_{\bar z} = \mu_n f_z$. Since each f_n is unique up to post-composition with a similarity transformation, we can normalize it so that it fixes 0 and 1. For instance, an explicit way to do so is to set

$$\tilde f_n(z) = \frac{f_n(z) - f_n(0)}{f_n(1) - f_n(0)},$$

then each $\tilde f_n$ still satisfies $f_{\bar z} = \mu_n f_z$ in the sense of distributions. By the preceding Good Approximation Lemma 6.24, there exists a quasiconformal homeomorphism f of $\mathbb{C}$ satisfying $f_{\bar z} = \mu f_z$ in the sense of distributions; it certainly fixes the points 0 and 1. By Corollary 6.5, f is unique. □ Theorem 6.3

Appendices

A Another Conformal Invariant

Let ρ denote a *density* — a nonnegative Borel-measurable function — on $\mathbb{R}^2$. Recall that the *ρ–length* of a path γ and the *ρ–area* of a Borel set E are defined as follows:

$$L_\rho(\gamma) = \int_\gamma \rho(s)\,|ds|, \qquad\qquad A_\rho(E) = \int_E \rho(x)^2\,dx.$$

A.1 Definitions. Let Γ be a path family in a planar domain Ω, that is, a collection of continuous maps from $\mathbb{R}$ to Ω. The *extremal length* of Γ in Ω is the extended real number given by

$$\mathrm{EL}_\Omega(\Gamma) = \sup_\rho \left\{ \frac{(\inf_{\gamma\in\Gamma} L_\rho(\gamma))^2}{A_\rho(\Omega)} : 0 < A_\rho(\Omega) < \infty \right\},$$

and the *conformal modulus* of Γ in Ω is defined by

$$\mathrm{M}_\Omega(\Gamma) = \inf_\rho \left\{ A_\rho(\Omega) \colon \inf_{\gamma\in\Gamma} L_\rho(\gamma) \geqslant 1 \right\}.$$

In other words, the (conformal) modulus is defined as the reciprocal of the extremal length after normalizing the class of admissible densities so that the ρ–length of every path in Γ is at least one.

In fact, the values of the extremal length and modulus only depend on the path family and not on the domain. Indeed, if $\Gamma \subset \Omega \subset \Omega'$, then every density ρ on Ω can be extended to a density ρ' on Ω' by setting $\rho'\big|_\Omega = \rho$ and $\rho'\big|_{\Omega'\setminus\Omega} \equiv 0$. This yields $\mathrm{EL}_\Omega(\Gamma) \leqslant \mathrm{EL}_{\Omega'}(\Gamma)$. Conversely, given a density ρ' on Ω', we define $\rho = \rho'\big|_\Omega$, which entails $\mathrm{EL}_\Omega(\Gamma) \geqslant \mathrm{EL}_{\Omega'}(\Gamma)$. For this reason, we will henceforth omit the subscripts in our notation for the extremal length and modulus.

A.2 Theorem. *Extremal length and modulus are conformally invariant.*

We shall only provide the proof for the extremal length; the claim about the modulus follows directly from its definition or by an identical argument.

Proof Let $\Gamma \subset \Omega$ be a path family and let f be conformal on Ω. Suppose that $\hat{\rho}$ is a density on $f(\Omega)$. We define a density $\rho(x) = (\hat{\rho} \circ f)(x) \, \|Df(x)\|$ on Ω and use change of variables $y = f(x)$ to calculate

$$L_\rho(\gamma) = \int_\gamma (\hat{\rho} \circ f)(x) \, \|Df(x)\| \, |dx| = \int_{f \circ \gamma} \hat{\rho}(y) \, |dy| = L_{\hat{\rho}}(f(\gamma)),$$
$$A_\rho(\Omega) = \int_\Omega |(\hat{\rho} \circ f)(x)|^2 \, \|Df(x)\|^2 \, dx = \int_{f(\Omega)} |\hat{\rho}(y)|^2 \, dy = A_{\hat{\rho}}(f(\Omega)).$$

The first line is mere change of variable $y = f(x)$ for integration on paths. In the second line, we use the identity from Theorem 1.9 (in view of conformality of f) together with the change of variables formula (5). We deduce that

$$\mathrm{EL}(\Gamma) \geqslant \mathrm{EL}(f(\Gamma)),$$

because $f(\Gamma) \subset f(\Omega)$. The inverse $f^{-1} \colon f(\Omega) \to \Omega$ is also conformal, so the reverse inequality holds by an identical argument. □

A.3 Examples of extremal length and conformal modulus. Let us illustrate the concepts with two very useful examples. These computations not only provide better understanding but also serve as a motivation for much of the following discussion.

Example 1: Rectangle Consider the rectangle $Q = (0, W) \times (0, H)$. Let $\Gamma_{Q,1}$ be the collection of rectifiable paths in Q which begin on the left edge and end on the right edge; similarly, let $\Gamma_{Q,2}$ consist of paths starting on the top edge and ending on the bottom edge. We claim that

$$\mathrm{EL}(\Gamma_{Q,1}) = \frac{W}{H} \qquad \text{and} \qquad \mathrm{EL}(\Gamma_{Q,2}) = \frac{H}{W}, \tag{A.34}$$

and, moreover, that

$$\mathrm{M}(\Gamma_{Q,1}) = \mathrm{EL}(\Gamma_{Q,2}) \qquad \text{and} \qquad \mathrm{M}(\Gamma_{Q,2}) = \mathrm{EL}(\Gamma_{Q,1}). \tag{A.35}$$

We only prove the first identity in (A.34); all the other ones are obtained in a completely analogous manner. We fix a height $y \in (0, H)$ and consider the horizontal path in $\Gamma_{Q,1}$ parametrized by $\gamma_y(t) = t + iy$ for $t \in [0, W]$. Integrating $L_\rho(\gamma_y)$ with respect to y, applying the Cauchy–Schwarz inequality and the Theorem of Fubini gives

$$H\, L_\rho(\gamma_y) = \int_0^H \int_0^W \rho(t+iy)\, dt\, dy \leqslant \left(\int_Q \rho^2(\zeta)\, d\zeta \right)^{\frac{1}{2}} \left(\int_0^H \int_0^W dt\, dy \right)^{\frac{1}{2}},$$

whence

$$\frac{(\inf_{\gamma\in\Gamma_1} L_\rho(\gamma))^2}{A_\rho(Q)} \leqslant \frac{W}{H}.$$

We obtain the reverse inequality by selecting a density $\rho \equiv 1$ on Q. This gives $A_\rho(Q) = WH$ and $L_\rho(\gamma) \geqslant W$ for every path $\gamma \in \Gamma_1$, thus completing the proof. □

EXAMPLE 2: ANNULUS Let $A = \mathcal{A}(r, R)$ be a nondegenerate annular domain and let $\Gamma_{A,1}$ be the family of paths in $\bar{A}$ which begin on $\mathbb{S}_r$ and end on $\mathbb{S}_R$. Consider also the family $\Gamma_{A,2}$ of closed loops in A whose winding number around the points of $\mathbb{D}_r$ at least one.

Rescaling the annulus obviously preserves the extremal length and modulus, so we may in fact replace A with $\mathcal{A}(1, R/r)$. Now, the complex logarithm maps the slit annulus $A \setminus [-R/r, -1]$ conformally onto the open rectangle $(0, \ln \frac{R}{r}) \times (-\pi, \pi)$, but the cut poses some topological complications. In order to avoid these issues, let us momentarily exclude the "problematic" paths: remove from $\Gamma_{A,1}$ those paths that meet the cut, and in $\Gamma_{A,2}$ consider only those paths that start and end on the same point of the cut and wind around the inner circle exactly once. We can see that the paths in the family $\Gamma_{A,1}$ are mapped by the logarithm to paths which connect the vertical edges, while the image of every path in $\Gamma_{A,2}$ separates the vertical edges of the rectangle. Thus, (A.34) and (A.35) imply

$$\mathrm{EL}(\Gamma_{A,1}) = \mathrm{M}(\Gamma_{A,2}) = \frac{1}{2\pi} \ln \frac{R}{r}, \tag{A.36}$$

$$\mathrm{EL}(\Gamma_{A,2}) = \mathrm{M}(\Gamma_{A,1}) = 2\pi \left(\ln \frac{R}{r} \right)^{-1}. \tag{A.37}$$

This derivation is intuitively clear, but it is not obvious whether we are justified in omitting the paths that meet the cut in the annulus from the computation. Luckily, it is possible to infer (A.36) and (A.37) with a direct argument analogous to that in the preceding example; we only need to switch to polar coordinates. Then, the upper bound for $\mathrm{EL}(\Gamma_{A,1})$ is again a consequence of the Cauchy–Schwarz inequality, while the lower bound is obtained by testing against a density $\rho(x) = |x|^{-1}$ on A. $\square$

A.4 Geometric interpretation. The geometric meaning of the extremal length and modulus is not the easiest thing to grasp.

The core reason why the problem of Grötzsch has no conformal solutions is the assertion of Theorem 1.9, which we used in passing from (1.7) to (1.8). Let us compare side-by-side the quantities

$$\frac{\|Df\|^2}{Jf} \qquad \text{and} \qquad \sup_{\rho} \frac{(\inf_{\gamma\in\Gamma} L_\rho(f\circ\gamma))^2}{A_\rho(f(\Omega))}$$

— the latter being the extremal length of a path family $f(\Gamma)$. We see immediately the analogy here: both expressions compare lengths and areas; both have special relationship with conformal mappings (Theorems 1.9 and A.2). A moment's thought reveals that the ratio on the left is in some sense the "infinitesimal" version of the extremal length — if we let Γ consist of paths that connect points on the boundary of an infinitesimal circle in Ω.

Further, relationships in (A.35) illustrate why some authors occasionally use the term *extremal thickness* or *width* for the modulus.

Lastly, formulæ (A.36) and (A.37) elucidate the tight connection between the the ring capacity and moduli of certain path families associated with rings.

B Quasiconformal Maps are Quasisymmetric

B.5 Alternative proof of Theorem 3.5 Once we have the knowledge that the inverse of a quasiconformal map is itself quasiconformal, we can expand the condition (3.11) to the following equivalent: f is a K–quasiconformal homeomorphism in Ω if there is a finite nonzero constant K such that the inequalities

$$\operatorname{Mod}(f(R)) \leqslant \frac{1}{K} \operatorname{Mod}(R), \tag{B.38}$$

$$\operatorname{Mod}(R) \leqslant \frac{1}{K} \operatorname{Mod}(f(R)) \tag{B.39}$$

hold simultaneously for every ring $R \subset\subset \Omega$ (see Remark 2.3). With the aid of property (B.39), we can now use the same method as in CASE 1 to resolve CASE 3 as follows.

CASE 3 We assume that $|x-y|/|x-z| \leqslant s$ where $0 < s < 1$.

Consider the ring S separating the segment $[x, y]$ from the circle $\mathbb{S}_{|x-z|}(x)$. Note that this ring is the circular inversion of the Grötzsch ring of radius $|x-z|/|x-y|$. The image of this ring, $S' = f(S)$, separates $\{f(x), f(y)\}$ from $\{f(z), \infty\}$. Similar to the previous case, Theorem 3.12, its Corollary 3.13, and condition (B.39) tell us that

$$\ln \Phi\left(\frac{|x-z|}{|x-y|}\right) = \operatorname{Mod}(S) \leqslant K \operatorname{Mod}(S') \leqslant K \ln \Psi\left(\frac{|f(x)-f(z)|}{|f(x)-f(y)|} + 1\right).$$

It now follows that

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \leqslant \frac{1}{\Psi^{-1}\left(\Phi\left(1\Big/\frac{|x-y|}{|x-z|}\right)^{\frac{1}{K}}\right) - 1}. \tag{B.40}$$

This implies that f is satisfies inequality (3.1) for every η dominating the function

$$t \mapsto \frac{1}{\Theta^{-1}\left(\frac{1}{t}\right)}, \qquad 0 < t < 1. \tag{B.41}$$

The properties of functions Φ and Ψ entail that the function defined by (B.41) is continuous and strictly increasing everywhere in $(0, 1)$. Moreover, it tends to zero when t does.

Finally, we observe that the function at (B.41) blows up as t approaches

1 from below, and no homeomorphism of $[0,\infty)$ can dominate it. However, this issue is resolved by our findings from CASE 2, in regards to the situation when $s < \frac{|x-y|}{|x-z|} < 1$, namely, we may use the notion of weak quasisymmetry.

B.6 Further remarks. There are several alternative strategies for proving the equivalence of the notions of quasisymmetry and quasiconformality. Although apparently dissimilar, all of these approaches express the same ideas using diverse mathematical vocabulary, and a side-by-side comparison is of interest in and of itself.

An interesting approach can be found in John Hubbard's book on *Teichmüller Theory* [Hub16]. There, a quasisymmetry is defined by associating to each triangle with vertices a_1, a_2, a_3 a quantity called the *skew*, defined by

$$\mathrm{Skew}(a_1, a_2, a_3) = \max \frac{|a_i - a_j|}{|a_i - a_k|}.$$

Naturally, the skew can be interpreted as the 'aspect ratio' of a triangle. Functions playing a similar role to that of Φ and Ψ are constructed in order to bound the skew in terms of the *annularity* of a triangle — the maximal ring module among all ring domains that separate one of the vertices from the other two. In Proposition 4.15.14, Hubbard shows that his definition of quasisymmetry is equivalent to our Definition 3.1; the proof involves constructing functions similar to those in (3.13) and (B.41). Interestingly, this proof reveals the same issue as the one we would face in CASE 3 if we let the parameter t approach one from below: the bounds blow up. The author of this thesis finds that the core limitation is that it is in general impossible to bound the ratio of the longest and shortest sides of a triangle in terms of the ratio of the longest and medium sides.

Note that the condition $0 < \frac{|x-y|}{|x-z|} < s$ for small values of s forces the triangle x, y, z to have large aspect ratio. This case is quite special; [Hub16, Lemma 4.5.15] tells us that that the shortest side of a tringle is mapped by a quasisymmetry to the shortest side of the image triangle if the skew of the image is greater than 3.

A more 'analytic' approach is taken in Astala, Iwaniec, and Martin's book [AIM08]. When proving that quasisymmetry implies quasiconformality in Section 3.4, the authors are working with the analytic, and make use of upper gradients and maximal inequalities. When establishing the partial converse, in Theorem 3.5.3, the case $\frac{|x-y|}{|x-z|} > 1$ is resolved by what can be called 'a module method in disguise,' using the Oscillation Lemma 3.5.1.

On a final note, the lecture notes [Kos09, Chapter 3] provide yet another proof of the same assertion by alluding to the metric definition $\mathrm{QC_M}$ and the local uniform boundedness of the distortion function $\frac{L_r f(x)}{\ell_r f(x)}$ and the egg-yolk principle.

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